Tolerance geometry – Euclid's first postulate for points and lines with extension

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ABSTRACT

Object representation and reasoning in vector based geographic information systems (GIS) is based on Euclidean geometry. Euclidean geometry is built upon Euclid's first postulate, stating that two points uniquely determine a line. This postulate makes geometric constructions unambiguous and thereby provides the foundation for consistent geometric reasoning. It holds for exact coordinate points and lines, but is violated, if points and lines are allowed to have extension. As an example for a point that has extension consider a point feature that represents the city of Vienna in a small scale GIS map representation. Geometric constructions with such a point feature easily produce inconsistencies in the data [23]. The present paper addresses the issue of consistency by formalizing Euclid's first postulate for geometric primitives that have extension.

We identify a list of six consequences from introducing extension: these are 'new qualities' that are not present in exact geometric reasoning, but must be taken into account when formalizing Euclid's first postulate for extended primitives. One important consequence is the positional tolerance of the incidence relation ("on"-relation). As another consequence, equality of geometric primitives becomes a matter of degree. To account for this fact, we propose a formalization of Euclid's first postulate in Łukasiewicz t-norm fuzzy logic. A model of the proposed formalization is given in the projective plane with elliptic metric. This is not a restriction, since the elliptic metric is locally Euclidean. We introduce graduated geometric reasoning with Rational Pavelka Logic as a means of approximating and propagating positional tolerance through the steps of a geometric construction process. Since the axioms (postulates) of geometry built upon one another, the proposed formalization of Euclid's first postulate provides the first building block of a geometric calculus that accounts for positional tolerance in a consistent way. The novel contribution of the paper is to define geometric reasoning with extended primitives as a calculus that propagates positional tolerance. Also new is the axiomatic approach to positional uncertainty and the associated consistency issue.

Categories and Subject Descriptors

I.2.3 [**Artificial Intelligence**]: Deduction and Theorem Proving - *uncertainty, "fuzzy," and probabilistic reasoning*.

General Terms

Theory.

Keywords

GIS, geometric reasoning, approximate reasoning, error propagation, positional tolerance, axiomatic geometry.

1. INTRODUCTION

In vector based geographic information systems (GIS) object representation is based on Euclidean geometry. Euclidean geometry relies on the idea that points are infinitely small. Points are the indivisible building blocks of geometric reasoning. In conflict with this idea is the fact that points in GIS map representations often represent geographic entities that in reality have extension. As an example, Figure 1 sketches two representations of the city of Vienna in different levels of detail: In Figure 1a, Vienna is represented by a single coordinate point, whereas in Figure 1b, its extended character is visible. A geometric construction that operates with the point representation of Vienna disregards its true extension, and consequently disregards the extension of the output. Existing heuristic solutions do not provide control over the behavior of a geometric construction w.r.t. extension. When plugged together, heuristics produce exceptions which must be treated separately. The present paper poses the question if it is possible to formalize geometric reasoning with points and lines that have extension *in a consistent way*.

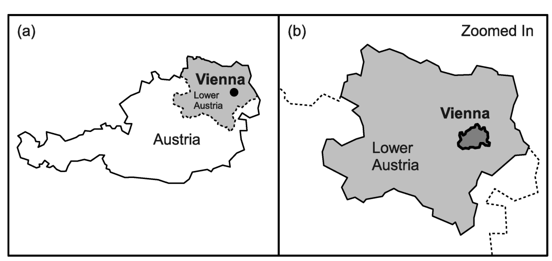

Figure 1. Vienna, (a) represented by a coordinate point, and (b) represented as an extended geographic entity.

We approach the issue of consistency by adopting an axiomatic standpoint of geometry. The paper provides a first step towards an answer of the above question by addressing the most fundamental axiom of all classical geometries, namely Euclid's first postulate. Euclid's first postulate states that the line determined by two points is unique. It makes geometric constructions unambiguous and thereby lays the foundation for consistent geometric reasoning.

As was shown in [23] positional tolerance plays a key role in geometric reasoning with extended primitives: An operator that connects two points with extension by a line with extension is either not practically useful in GIS, or introduces ambiguity into Euclid's first postulate: There is no law of nature telling the 'right' way of connecting Vienna with Munich by a linear entity. Despite this lack of principle, the generated ambiguities are not arbitrary: Based on some general assumptions on the nature of extended objects in a GIS context, we show that it is possible to derive a location constraint for the output object of a geometric construction. Since location constraints provide a certain amount of tolerance in positioning an object, we call the resulting ambiguity 'positional tolerance', and the respective formalism 'tolerance geometry'.

As a consequence of introducing extension, six 'new qualities' emerge in connection with Euclid's first postulate: positional tolerance of incidence, significance of size and distance, graduation of equality of points and of lines, intransitivity of equality, and granularity. We provide a formalization of Euclid's first postulate for extended primitives that accounts for these 'new qualities'. One way to define a graduated version of the equality relation is to define a reasonable fuzzy extension. The proposed formalization of Euclid's first postulate extends the Boolean formalization by translating it into fuzzy logic, and adapting it to fit the granularity component. Due to its metric properties, we choose Łukasiewicz t-norm fuzzy logic. We provide a model of the proposed fuzzy formalization in the projective plane with the elliptic metric. Since the elliptic plane is locally Euclidean, the proposed model is an approximation for Euclidean reasoning.

As a means of approximating and propagating positional tolerance, we introduce graduated geometric reasoning with Rational Pavelka Logic. Rational Pavelka Logic builds upon Łukasiewicz fuzzy logic, and allows for deriving a degree of equality of extended lines from Euclid's first postulate.

The remainder of the paper is structured as follows: Chapter 2 gives a review of related work. Chapter 3 discusses general assumptions on points and lines with extension. We derive a list six 'new qualities' that result from introducing extended primitives into Euclid's first postulate. Chapter 4 develops formalizations and interpretations of the incidence and equality relations for extended primitives that are based on the six new qualities. Chapter 5 gives a fuzzification of Euclid's first postulate in Łukasiewicz logic. Chapter 6 introduces graduated geometric reasoning with Rational Pavelka Logic. Chapter 7 concludes with a summary of contributions and chapter 8 gives an outlook to future work.

2. RELATED WORK

Extended objects may be interpreted as location constraints. The issue of geometric reasoning with extended objects can be seen as a special case of geometric reasoning with positional uncertainty. Some of the oldest references addressing this topic are [16], [18], or [3]. Among others, these authors developed the concept of epsilon tolerance and fuzzy tolerance to cope with the problem of spurious objects that result from positional uncertainty. The approaches improved on a multitude of different problems concerning e.g. coincidence of points, line crossing and conflation. Yet, a closed and consistent solution for all geometric constellations and operations is still missing. The present work addresses the issue of consistency by the attempt to integrate the concept of positional tolerance into the axiomatic foundation of geometric reasoning. As a first step towards this goal, the present paper formalizes the most fundamental axiom of geometry, Euclid's first postulate, in mathematical fuzzy logic. The axiomatic approach has three advantages: First, it is structurally close to exact geometry. Second, consistency of the calculus can be investigated with the tools of mathematical logic. Third, it allows for logical deduction and theorem proving.

Numerous axiomatic approaches exist, that aim at restoring classical geometry from primitives that have extension. These approaches are inspired by a similar idea as the present work, namely the idea that extensionless points and lines are abstractions, and as such do not exist in reality. In contrast to the present work, their aim is to restore exact geometry from extended primitives, and not to approximate the behavior of exact geometry with extended primitives. The best known of these approaches is A. Tarski's Geometry of Solids [21], followed by H. J. Schmidt [19], and, G. Gerla [9], [10]. In the GIS community B. Bennett's 'Region Based Geometry' [1] is of particular importance.

In the present paper, we use mathematical fuzzy logic to fuzzify the process of geometric *reasoning*, but do not use fuzzy logic for object description: Vague geographical objects, and operations on them, can be described with the tools of fuzzy set theory (e.g., [7], [8], [6], [5]). In contrast to this, the present paper uses t-norm fuzzy logic [12] which is a branch of mathematical logic and developed from multivalued logical systems.

Recent approaches to geometric reasoning with positional uncertainty in GIS are mostly concerned with positional uncertainty that is caused by stochastic variability. Y. Leung et. al. [14] proposed a probabilistic framework for error analysis in measurement based GIS. Another example stems from the field of computer vision: Under the title of 'Uncertain Projective Geometry' S. Heuel [13] proposes a projective geometric calculus for points and lines that have positional uncertainty. Heuel augments the formalism of Grassmann-Caley algebra by attaching to each exact projective point and exact projective line, respectively, the covariance matrix of a Gaussian probability density function. As a result, an 'uncertain point' is described by a pair (*exact point, Gaussian*); an 'uncertain line' is described by a pair (*exact line, Gaussian*). In a geometric construction step, the exact formalism is applied to the first entry, while the law of error propagation is applied to the covariance matrix of the second entry. The advantage of Heuel's calculus lies in the good approximation properties of error propagation for Gaussians over linear functions. This advantage vanishes if general probability density functions are used. Consequently the theory cannot be applied to extended objects, which, in case they stem from limited measurement resolution or lack of information, would have to be described by uniform distributions.

Another recent reference is E. Clementini [4], who gives a model for GIS line features that have extension. His 'model for uncertain lines' is built upon the model for regions with broad boundaries and allows for deriving topological relations between 'uncertain lines', but does not discuss geometrical operations.

3. EXTENDED GEOMETRIC PRIMITIVES

In the following subchapter 3.1 we motivate, from a GIS perspective, a reasonable definition of 'point with extension' and 'line with extension'. Subchapters 3.2 - 3.6 introduce a total of six 'new qualities' that have to be taken into account when considering extended primitives in conjunction with Euclid's first postulate. These new qualities are the following:

1) The incidence relation between extended points and extended lines is subject to positional tolerance.
2) Size and distance matter.
3) Equality of extended lines is a matter of degree.
4) Granularity is introduced due to varying object sizes.
5) Equality of extended points is a matter of degree.
6) Equality of extended objects is intransitive.

3.1 General Assumptions on Points and Lines with Extension

In the present paper we are concerned with entities that have extension in geographic space, and whose GIS representations are intended to be used as primitives of a geometric construction.

We differentiate between an *actual* GIS object representation, and a *potential* representation as an extended object in an underlying metric space. As shown in the introduction using the example of Vienna, the actual GIS object representation can be punctual, while potentially the objects' extension can be displayed, e.g. in a different level of detail. In other words, a metric space exists, where the object's extension is representable.

For object representation in GIS it is common to use either Cartesian coordinates or homogeneous coordinates. Homogeneous coordinates are coordinates for the projective plane. The usual metric used is the Euclidean metric in a Cartesian vector space and the elliptic metric in the projective plane. Since the elliptic plane is locally Euclidean, elliptic distance can be used to approximate Euclidean distance locally [2].

In the present paper we confine our considerations to two dimensional object representations. We further confine ourselves to extended geographic entities with sharp ("crisp") boundaries: For such objects, the indeterminacy of the object's boundary is negligible compared to its extension in space. Examples are parcels of land, buildings, countries, most lakes, roads, or streams. We do not consider point measurements like GPS coordinate points, which might have relevant measurement inaccuracy. We also do not consider objects with vague boundaries like mountains or pollution plumes.

Extended geographic objects can be classified as 'punctual' or 'linear', depending on context and grain size (e.g. [20]), or depending on semantics. The present paper assumes no such classification. Instead, the notions 'point with extension' and 'line with extension' are derived from points and lines as they are used in Euclidean and projective axiomatic geometry. Here, points and lines are different sorts of objects, complying with a set of axioms, namely the axioms of Euclidean or projective geometry. With the terms "extended point" and "extended line" we refer to an augmentation of a specific model of Euclidean or projective geometry, in which exact points and lines have an interpretation. As an example, the Cartesian interpretation of a two dimensional Euclidean point is a pair of real numbers. We can 'add extension' to the point by considering a topological neighborhood of the point in the real plane. A Cartesian line in the plane can be represented by a tupel of parameter, e.g. an incident Cartesian point, and direction. We can 'add extension' to a Cartesian line by considering a topological neighborhood in its parameter space.

Using homogeneous coordinates simplifies the representation task for extended points and lines: Exact projective points and lines can both be represented by vectors of the two dimensional unit sphere $\mathbb{S}^2 := \left\{ p \in \mathbb{R}^3 \mid \|p\| = 1 \right\}$, where antipodal points, $+p$ and $-p$ are identified: $\mathbb{P}^2 := \mathbb{S}^2 / \pm$. Consequently the parameter space of an exact elliptic line is identical to the parameter space of an exact elliptic point. Using the duality of exact points and lines, it is sufficient to define extended points and derive a definition of extended lines from duality:

Definition 1. An *extended elliptic point* is a non-empty, bounded and regular closed subset of the elliptic space. A set of exact projective lines in the elliptic plane is an *extended line*, if the set of dual exact points is an extended point.

As mentioned above, the canonical metric on the projective plane $\mathbb{P}^2 = \mathbb{S}^2 / \pm$ is the elliptic metric

$$\varepsilon : \mathbb{P}^2 \to [0, \pi/2], \quad \varepsilon(p,q) := \operatorname{Arccos}|p \cdot q|, \tag{1}$$

where $p \cdot q = p_1 q_1 + p_2 q_2 + p_3 q_3$ denotes the standard inner product in $\mathbb{R}^3$. The term 'regular closed' in definition 1 refers to the topology on $\mathbb{P}^2$, which is induced by ε. Note that in the elliptic plane, exact lines are unbounded, but have finite length π. The maximum distance of any two points is $\pi/2$. This is not a restriction, since we are interested in bounded map representations, and do not consider the global behavior of the geometry. Every non-zero scaling of ε is also a metric distance on $\mathbb{P}^2$. In particular $(2/\pi) \cdot \varepsilon$ is an elliptic metric, where the maximum distance of any two points is normalized to 1 [2].

3.2 Incidence of Extended Primitives has Positional Tolerance

As already mentioned in the forgoing subchapter, axiomatic geometry is an abstract logical theory. An interpretation of such a theory in another domain, like the Cartesian interpretation in the real plane, is called a model of the theory, if it complies with its axioms. As a first step towards a geometric theory of extended primitives, the present paper is concerned with only one axiom, namely Euclid's first postulate, which is a part of all classical geometries, in particular of Euclidean and projective geometry. It was shown in [23] that a model of Euclid's first postulate that operates with extended primitives must hold object shape and object size fixed, and consequently does not carry any information about an object's extension. As a consequence, it is necessary to extend Euclid's first postulate and it's interpretations.

Figure 2 sketches three typical geographic scenarios, where two extended objects can be classified as incident: Figure 2a shows a floating water mill that is installed in a stream. The polygon representing the mill can be interpreted as an extended point; the stream can be interpreted as (a segment of) an extended line. The polygon representing the mill is a proper part of the polygon representing the stream. Figure 2b shows a station building of a city railway and its tracks. Clearly, the building overlaps with the tracks. Figure 2c shows a marina on the banks of stream. If the marina is aggregated to a single polygon, it clearly overlaps the polygon representing the stream. These examples suggest that incidence of extended objects can be modeled by the *overlap* relation in the underlying metric space.

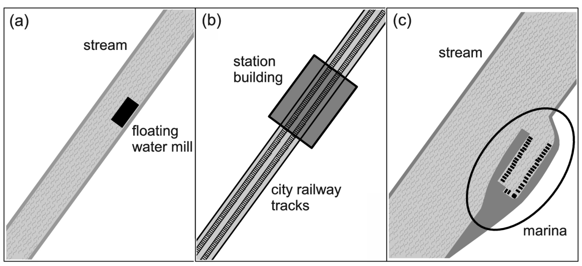

Figure 2. Three examples of extended objects that are incident with an extended line.

Note that in axiomatic geometry, the notion of 'line segment' is derived from the more general notion of 'line', after an order relation ('betweenness') has been added to the theory. It is not possible to give a real world example of an extended line, just as it is not possible to give a real world example of an infinitely long straight exact line. This fact does not disqualify the motivating examples.

3.3 Equality of Extended Lines is Graduated

The location of the extended points creates a constraint on the location of an incident extended line. In exact geometry this location constraint fixes the position of the line uniquely. In case points and lines are allowed to have extension this is not the case. Consequently Euclid's First postulate does not apply: Figure 3 shows that if two distinct extended points P and Q are incident (i.e. overlap) with two extended lines L and M, then L and M are not necessarily equal.

Yet, in most cases, L and M are "closer together", i.e. "more equal" than arbitrary extended lines that have only one or no extended point in common. The further P and Q move apart from each other, the more similar L and M become. One way to model this fact is to allow *degrees of equality* for extended lines. In other words, the equality relation is graduated: It allows not only for Boolean values, but for values in the whole interval $[0,1]$.

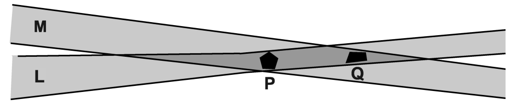

Figure 3. Two extended points do not uniquely determine the location of an incident extended line.

3.4 Size and distance matters

In exact coordinate geometry, points and lines do not have size. As a consequence, distance of points does not matter in the formulation of Euclid's first postulate[1]. If points and lines are allowed to have extension, both, size and distance matter.

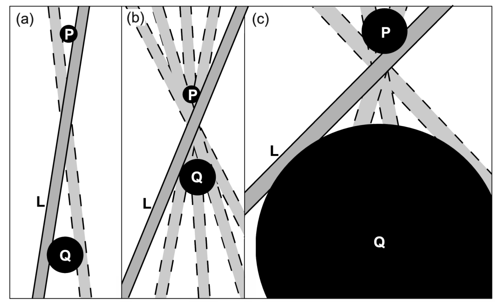

Figure 4. Size and distance matter.

Figure 4 depicts the location constraint on an extended line L that is incident with the extended points P an Q. The location constraint can be interpreted as *tolerance in the position of* L. In Figure 4a the distance of P and Q is large with respect to the sizes of P and Q, and with respect to the width of L. The resulting positional tolerance for L is "small". In Figure 4b, the distance of P and Q is smaller than it is in Figure 4a. As a consequence the positional tolerance for L becomes larger. In Figure 4c, P and Q have the same distance than in Figure 4a, but their sizes are increased. Again, positional tolerance of L increases.

As a consequence, a formalization of Euclid's first postulate for extended primitives must take all three parameters into account: the distance of the extended points, their size, and the size of the incident line. In the present paper we simplify the formalizing task by *limiting the scope of the incidence relation*: instead of considering an overlap relation, as suggested in chapter 3.2, we model incidence by the *subset relation*. In other words, an extended line can not be incident with a point that is 'larger' than itself. In Figure 2, only example (a) of the floating water mill complies with this interpretation.

3.5 Extension Introduces Granularity

In exact coordinate geometry, two distinct coordinate points p and q determine (the location of) a coordinate line uniquely, even if p and q are arbitrarily close to one another. This is not necessarily the case for distinct extended points P and Q. To see this, imagine that the extended points P and Q in Figure 3 move closer together: If P and Q are "very close" to one another and the extended line L is "too broad", then it may

[1] More specifically, the discrete metric Δ is used: $\Delta(p,q)=1$, if $p \neq q$, and $\Delta(p,q)=0$, if $p=q$.

happen that P and Q behave like one single point with respect to L.

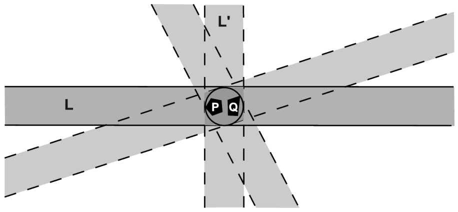

Figure 5. P and Q are indiscernible for L.

Figure 5 illustrates this case: Despite the fact that P and Q are distinct extended points that are both incident with L, they do not specify any *directional constraint* for L. Consequently, the directional parameter of the extended lines L and L' in Figure 5 may assume its maximum (at 90°). If we measure similarity (i.e. graduated equality) as inverse to distance, and if we establish a distance measure between extended lines that depends on all parameters of the line's parameter space, then L and L' in Figure 5 must have *maximum* distance. In other words, their degree of equality is zero, even though they are distinct and incident with P and Q.

The above observation can be interpreted as *granularity*: If we interpret the extended line L in Figure 5 "as a sensor", then the extended points P and Q are "indiscernible for L". Note that in this context grain size is not constant, but depends on the line that serves "as a sensor".

3.6 Equality of Extended Points is Graduated

Graduated equality of extended lines compels graduated equality of extended points. Figure 5a sketches a situation where two extended lines L and M intersect in an extended point P. If a third extended line L' is "very similar" to L, its intersection with M yields an extended point P' which is "very similar" to P. It is desirable to model this fact. To do so, it is necessary to allow graduated equality of extended points.

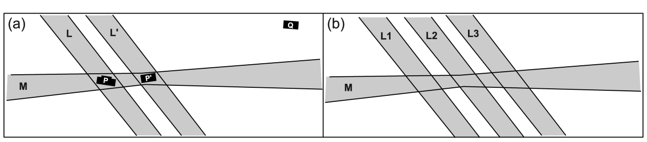

Figure 5. (a) Graduated equality of extended lines compels graduated equality of extended points. (b) Equality of extended lines is not transitive.

3.7 Graduated Equality is not Transitive

Figure 5b illustrates that an equality relation between extended objects need not be transitive. This phenomenon is commonly referred to as the Poincaré paradox. The Poincaré paradox is named after the famous french mathematician and theoretical physicist Henri Poincaré, who repeatedly pointed this fact out, e.g. in [17], referring to indiscernibility in sensations and measurements. Note that this phenomenon is usually insignificant, if positional uncertainty is caused by stochastic variability. In measurements, the stochastic variability caused by measurement inaccuracy is usually much grater than the indiscernibility caused by limited resolution. For extended objects, this relation is reversed: The extension of an object can be interpreted as indiscernibility of its contributing points. As stated in chapter 3.1, the present paper starts off from the assumption that the extension of an object is big compared with the indeterminacy of its boundary.

4. EQUALITY AND INCIDENCE OF EXTENDED POINTS AND LINES

In subchapter 4.1 we propose an interpretation of the incidence predicate for extended points and lines that have positional tolerance (cf. chapters 3.2 and 3.4). Chapter 4.2 develops a formalization of graduated equality predicates for extended points and lines: in a step by step manner, we include the graduation property (cf. chapters 3.3 and 3.6), the Poincaré paradox (cf. chapter 3.7), and granularity (cf. chapter 3.5).

4.1 Incidence of extended points and lines

As shown in chapter 3.2, it is a reasonable assumption in the GIS context to classify an extended point and an extended line as incident, if their extended representations in the underlying metric space overlap. In chapter 3.4 we argued that we intend to limit the scope of the incidence relation in this paper in order to simplify the formalism. We do this by modeling incidence by the subset relation:

Definition 2: For an extended point P, and an extended line L we define the incidence relation by

$$On(P,L) := \left(P \subseteq L\right) \quad \in \{0,1\}, \tag{2}$$

where the subset relation $\subseteq$ refers to P and L as subsets of the underlying metric space.

The extended incidence relation (2) is a Boolean relation, assuming either the truth value 1 (*true*) or the truth value 0 (*false*). Since a Boolean relation is a special case of a graduated relation, i.e. since $\{0,1\} \subset [0,1]$, we will be able to use relation (2) as part of a fuzzified Euclid's first postulate later on. The interpretation of the subset relation (2) in the elliptic plane is straight forward.

4.2 Equality of extended points and lines

As stated in chapters 3.3 and 3.6, equality of extended points, and equality of extended lines is a matter of degree. Geometric reasoning with extended points and extended lines relies heavily on the metric structure of the underlying coordinate space. Consequently, it is reasonable to model graduated equality as inverse to distance.

4.2.1 Metric Distance

A pseudometric distance, or *pseudometric*, is a map $d: M^2 \to \mathbb{R}^+$ from a domain M into the positive real numbers (including zero), which is minimal, symmetric, and satisfies the triangle inequality:

$$d(x,x) = 0 \tag{3}$$

$$d(x,y) = d(y,x) \tag{4}$$

$$d(x,y) + d(y,z) \geq d(x,z). \tag{5}$$

d is called a *metric*, if additionally separability holds:

$$d(x,y)=0 \iff x=y. \tag{6}$$

Well known examples of metric distances are the Euclidean distance, or the Manhattan distance. Another example is the elliptic metric for the projective plane defined in (1).

The "upside-down-version" of a pseudometric distance is a *fuzzy equivalence relation w.r.t. the Łukasiewicz t-norm*. The next chapter briefly introduces the logical connectives in Łukasiewicz t-norm fuzzy logic. We will use this particular fuzzy logic to formalize Euclid's first postulate for extended primitives in chapter 5. The reason for choosing Łukasiewicz fuzzy logic is its strong connection to metric distance.

4.2.2 Łukasiewicz Logic

Łukasiewicz logic is one of the three fundamental t-norm fuzzy logics. In t-norm fuzzy logics, a *triangular norm (t-norm)* plays the role of a graduated conjunction operator. A t-norm is a binary operation $*:[0,1]^2 \to [0,1]$ that is commutative, associative, non-decreasing, and has 1 as its unit element [12]. For the fuzzification of Euclid's first postulate in chapter 5, we will additionally need an implication and a negation operator. The *Łukasiewicz t-norm* $\otimes$, its residuated implication $\to$, and the corresponding negation $\neg$ are given by

$$x \otimes y = \max\{x+y-1,0\}, \tag{7}$$

$$x \to y = \begin{cases} 1 & \text{for } x \le y \\ 1-x+y & \text{for } x > y \end{cases}, \tag{8}$$

$$(\neg x) = (1-x), \tag{9}$$

respectively. In chapter 6.2, we will need the following equivalences:

$$[x \to y]=1 \iff x \le y, \tag{10}$$

$$[(x \otimes z) \to y]=1 \iff [z \to (x \to y)]=1. \tag{11}$$

4.2.3 Fuzzy Equivalence Relations

As mentioned above, the "upside-down-version" of a pseudometric distance is a *fuzzy equivalence relation w.r.t. the Lukasiewicz t-norm* $\otimes$. A fuzzy equivalence relation w.r.t. $\otimes$ is a fuzzy relation $e: M^2 \to [0,1]$ on a domain M, which is reflexive, symmetric and $\otimes$-transitive:

$$e(x,x)=1 \tag{12}$$

$$e(x,y)=e(y,x) \tag{13}$$

$$e(x,y)\otimes e(y,z) \le e(x,z). \tag{14}$$

e is called a *fuzzy equality relation*, if additionally separability holds:

$$e(x,y)=1 \iff x=y. \tag{15}$$

If d is a pseudometric distance, then

$$e(x,y) := \max\{1-d(x,y),0\} \tag{16}$$

is a fuzzy equivalence relation w.r.t. the Łukasiewicz t-norm $\otimes$ [22]. In case the size of the domain M is normalized to 1, equation (16) simplifies to

$$e(x,y) := 1-d(x,y). \tag{17}$$

In other words, given a metric distance d on a normalized domain, equation (17) defines a graduated equality relation e by simple Łukasiewicz negation.

As stated in chapter 3.6, it is necessary to account for the Poincaré paradox when defining a graduated equality relation between extended points and lines. The next chapter shows how the Poincaré paradox can be integrated in the formalism of fuzzy equivalence relations without dropping transitivity completely. It is not desirable to drop transitivity of equality, since it is the only ingredient that enables us to relate two extended objects via a third [25].

4.2.4 Approximate Fuzzy Equivalence Relations

G. Gerla (2008) shows that for modeling the Poincaré paradox in a graduated context transitivity may be replaced by a weaker form:

$$e(x,y)\otimes e(y,z)\otimes dis(y) \le e(x,z). \tag{18}$$

Here $dis: M \to [0,1]$ is a lower-bound measure for the degree of transitivity that is permitted by y. A pair (e, dis) that is reflexive (12), symmetric (13), and weakly transitive (18) is called an *approximate fuzzy* $\otimes$ *-equivalence relation*[2].

In the setting of Euclid's first postulate shown in Figure 3 (chapter 3.3), the only link between the extended lines L and M are the extended points P and Q. Approximate fuzzy equivalence relations allow determining the equality degree of L and M via the *extended connection*

$$\overline{PQ} := \{l_{(p,q)} \mid p \in P, q \in Q, p \ne q\} \tag{19}$$

of P and Q. Here, $l_{(p,q)}$ is the unique exact line connecting the exact points p and q. Figure 6a sketches $\overline{PQ}$ in the Euclidean plane. Figures 6b and 6c show two arbitrary extended lines L and M that are incident with P and Q.

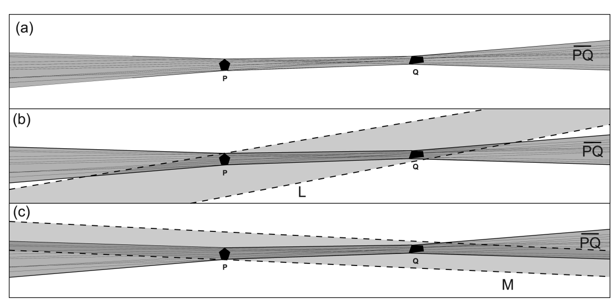

Figure 6. The equality of L and M can be determined via $\overline{PQ}$.

[2] Gerla uses the name *approximate similarity relation*. In the present abstract we use the name *approximate fuzzy equivalence relation* to stress the connection with the Boolean equality relation used in Euclid's first postulate.

An approximate fuzzy $\otimes$-equivalence relation is the "upside-down-version" of a so-called *pointless pseudometric space* (δ, s):

$$\delta(x,x)=1, \tag{20}$$

$$\delta(x,y)=\delta(y,x), \tag{21}$$

$$\delta(x,y)+\delta(y,z)+s(y)\geq\delta(x,z). \tag{22}$$

Here, $\delta: M\to\mathbb{R}^{+}$ is a (not necessarily metric) distance between extended regions, and $s: M\to\mathbb{R}^{+}$ is a size measure. Inequality (22) is a weak form of the triangle inequality. It corresponds to the weak transitivity (18) of the approximate fuzzy $\otimes$-equivalence relation e.

In case the size of the domain M is normalized to 1, e and dis can be represented by [11]

$$e(x,y):=1-\delta(x,y),\quad dis(y):=1-s(y)\,. \tag{23}$$

In other words, given a pointless pseudometric (δ, s) for extended regions on a normalized domain, equations (23) define an approximate fuzzy $\otimes$-equivalence relation *(e,dis)* by simple Łukasiewicz negation. The so defined equivalence relation on the one hand complies with the Poincaré paradox, and on the other hand retains enough information to link two extended points (or lines) via a third.

An example of a pointless pseudometric space is the set of extended points in the elliptic plane with the following measures [11]:

$$\delta(P,Q):=\inf\left\{d(p,q)\mid p\in P, q\in Q\right\}, \tag{24}$$

$$s(P):=\sup\left\{d(p,q)\mid p,q\in P\right\}, \tag{25}$$

where $d:=\left[(2/\pi)\cdot\varepsilon\right]:\mathbb{P}^{2}\to[0,1]$ is the normalized elliptic metric defined in chapter 3.1. A pointless metric distance on the set of elliptic extended lines can be defined in the dual space:

$$\delta(L,M):=\inf\left\{d(l',m')\mid l\in L, m\in M\right\}, \tag{26}$$

$$s(L):=\sup\left\{d(l',m')\mid l,m\in L\right\}. \tag{27}$$

4.2.5 Boundary Conditions for Granularity

As discussed in chapter 3.4 granularity enters Euclid's first postulate, if points and lines have extension: If two extended points *P* and *Q* are "too close" and the extended line *L* is "too broad", then *P* and *Q* are indiscernible for *L*. Since this relation of indicernibility (equality) depends not only on *P* and *Q*, but also on the extended line *L*, which acts "as a sensor", we denote it by $e(P,Q)[L]$, where *L* serves as an additional parameter for the equality of *P* and *Q*.

We propose three boundary conditions to specify a reasonable behavior of $e(P,Q)[L]$:

1. If $s(L)\geq\delta(P,Q)+s(P)+s(Q)$, then P and Q impose no direction constraint on L (cf. Figure 5), i.e. *P* and *Q* are "indiscernible for *L*" to degree 1: $e(P,Q)[L]=1$.
2. If $s(L)<\delta(P,Q)+s(P)+s(Q)$, then P and Q impose *some* direction constraint on L, but in general do not fix its location unambiguously. Accordingly, the degree of indiscernibility of P and Q lies between zero and one: $0<e(P,Q)[L]<1$.
3. If $s(L)<\delta(P,Q)+s(P)+s(Q)$ and $P=p$, $Q=q$ and $L=l$ are crisp, then $s(L)=s(P)=s(Q)=0$. Consequently, p and q determine the direction of l unambiguously, and all positional tolerance disappears. For this case we demand $e(P,Q)[L]=0$.

4.2.6 An Elliptic Model of Granulated Equality

From the boundary conditions on granularity proposed in the forgoing subchapter, we can derive an interpretation of granulated equality in the elliptic plane. Unfortunately the derivation is too long for presentation in the present paper. Instead, we make it available on the authors' website [24].

Definition 4. For extended elliptic points *P*, *Q*, and an extended elliptic line *L*, we define the *elliptic granulated equality relation* as follows:

$$e(P,Q)[L]:=\min\left\{(2/\pi)\,\mathrm{Arc}\sin\left(H(P,Q)[L]\right),\ 1\right\} \tag{28}$$

where

$$H(P,Q)[L]:=\begin{cases}h(P,Q)[L], & \text{if}\quad \delta(P,Q)+s(P)+s(Q)\neq 0,\\ \infty, & \text{if}\quad \delta(P,Q)+s(P)+s(Q)=0,\end{cases} \tag{29}$$

and

$$h(P,Q)[L]:=\frac{\tan\left[(\pi/4)\cdot s(L)\right]}{\tan\left[(\pi/4)\cdot\left(\delta(P,Q)+s(P)+s(Q)\right)\right]}\,. \tag{30}$$

Here, $\delta(P,Q)$, $s(P)$, $s(Q)$ and $s(L)$ are specified by definitions (24), (25) and (27).

5. A FUZZIFICATION OF EUCLID'S FIRST POSTULATE

In chapters 3 and 4 we identified and formalized a number of new qualities that enter into Euclid's first postulate, if extended geometric primitives are assumed. Consequently, we are now in the position of formulating a fuzzified version of Euclid's first postulate. To do this, we first split the postulate

"Two distinct points determine a line uniquely." (31)

into two subsentences

"Given two distinct points, there exists at least one line that passes through them. " (32)

"If more than one line passes through them, then they are equal." (33)

These subsentences can be formalized in Boolean predicate logic as follows:

$$\forall p,q\,.\,\exists l\,.\left[on(p,l)\wedge on(q,l)\right], \tag{34}$$

$$\forall p,q,l,m\,.\left[\neg(p=q)\right]\wedge\left[on(p,l)\ \wedge\ on(q,l)\right] \wedge\left[on(p,m)\ \wedge\ on(q,m)\right]\rightarrow\ (l=m). \tag{35}$$

Here, p, q stand for (exact) points, and l, m stand for (exact) lines. The Boolean predicates = and *on* refer to equality and incidence. $\forall$ and $\exists$ denote the universal and existential quantifiers, $\neg,\wedge,\rightarrow$ stand for Boolean negation, conjunction, and implication, respectively. In the remainder of this paper, we refer to (34) as the *exact existence property*, and to (35) as the *exact uniqueness property* of incidence geometry.

A verbatim translation of (34) and (35) in the syntax of Łukasiewicz fuzzy logic yields

$$\inf_{P,Q}\sup_{L}\left[on(P,L)\wedge on(Q,L)\right], \tag{36}$$

$$\inf_{P,Q,L,M}\left\{\left[\neg e(P,Q)\right]\ \otimes\ \left[on(P,L)\otimes on(Q,L)\right] \otimes\left[on(P,M)\otimes on(Q,M)\right]\ \rightarrow\ e(L,M)\right\}, \tag{37}$$

where P, Q denote extended points, L, M denote extended lines, and the symbol e replaces the symbol = for equality, since = is usually reserved to denote Boolean equality. The universal quantifier is replaced by the infimum operator; the existential quantifier is replaced by the supremum operator [12].

The translated existence property (36) can be adopted as it is, but the translated uniqueness property (37) must be adapted to include granulated equality of extended points:

In contrast to the Boolean case, the degree of equality of two given extended points is not constant, but depends on the extended line that acts "as a sensor". Consequently, the term $\neg e(P,Q)$ on the left hand side of (37) must be replaced by two terms, $\neg e(P,Q)[L]$ and $\neg e(P,Q)[M]$, one for each line, L and M, respectively:

$$\inf_{P,Q,L,M}\left\{\left[\neg e(P,Q)[L]\ \otimes\ \neg e(P,Q)[M]\right] \otimes\ \left[on(P,L)\ \otimes\ on(Q,L)\right] \otimes\ \left[on(P,M)\ \otimes\ on(Q,M)\right]\ \rightarrow\ e(L,M)\right\}. \tag{38}$$

In order to use weak transitivity of graduated equality, the discernibility measure of the extended connection $\overline{PQ}$ must be added:

$$\inf_{P,Q,L,M}\left\{\left[\neg e(P,Q)[L]\ \otimes\ \neg e(P,Q)[M]\otimes dis(\overline{PQ})\right] \otimes\ \left[on(P,L)\ \otimes\ on(Q,L)\right] \otimes\ \left[on(P,M)\ \otimes\ on(Q,M)\right]\ \rightarrow\ e(L,M)\right\}. \tag{39}$$

In the following, we refer to (36) as the *graduated existence property*, and to (38) as the *graduated uniqueness property*.

The graduated existance property (36) can be formulated in words as follows:

"Given two extended points, there is always an extended line that overlaps with them." (40)

The graduated uniqueness property (38) can be formulated in words as follows:

"Two (more or less) distinct extended points determine an extended line up to (more or less great) positional tolerance." (41)

These sloppy formulations reflect the fact that, given two *suitable* extended points, it is possible (for humans) to draw an approximate extended line that connects them. Properties (36) and (38) formalize this effect: The granulated equality predicates $e(P,Q)[L]$ and $e(P,Q)[M]$ in (38) measure "how suitable" P and Q are. $e(L,M)$ expresses a worst case measure of "how equal" two extended lines L and M are, provided they both overlap with P and Q.

If we use only one line L as an input to the graduated uniqueness property (38), i.e. if $L=M$, then $e(L,M)$ is a measure of how much positional tolerance L has, if it is "hooked up" to P and Q, i.e. if P and Q are subsets of L.

It is possible to prove that the elliptic interpretations of incidence given by (2), graduated equality of extended lines given by (23) and (26), and granulated equality of extended points given by (28) define a model of the proposed fuzzifications (36) and (38) of Euclid's first postulate. Unfortunately, the length restriction of the present paper format does not allow for presenting it here. Instead, we provide the proof on the author's website [24].

6. GRADUATED GEOMETRIC REASONING

In this last chapter we address the question of how the proposed formalization (36) and (38) of an "extended version" of Euclid's first postulate can be used in practical applications. Chapter 6.1 introduces Rational Pavelka Logic (RPL) as a tool for graduated reasoning. Chapter 6.2 shows how RPL can be used to propagate estimates of the positional tolerance through the steps of a geometric construction process.

6.1 Rational Pavelka Logic

As stated in chapter 4.2, one reason for choosing Łukasiewicz logic is its strong connection to metric spaces. Another reason is that Łukasiewicz logic can be extended to allow for graduated deduction rules in the style of J. Pavelka ([12], [15]). The according logic is called Rational Pavelka Logic (RPL). Graduated deduction rules allow for inferring partially true conclusions from partially true assumptions. This is useful in our setting: We can interpret the degree $\lambda<1$ of equality of two extended lines L and M by saying that the proposition "L and M are equal" is partially true, with truth degree λ.

As the most important deduction rule, the exact Modus Ponens rule is given by

$$\frac{A,A\rightarrow B}{B}. \tag{42}$$

The formulas above the line are *assumptions (facts)*, and the formula below the line is the *conclusion* of the rule. The exact Modus Ponens deduces true conclusions from true assumptions: If and $A\rightarrow B$ have truth degree 1, then B has truth degree 1.

The graduated modus ponens rule is given by

$$\frac{(A,\mu_A),(A\rightarrow B,\mu_{A\rightarrow B})}{(B,\mu_A\otimes\mu_{A\rightarrow B})}, \tag{43}$$

where μ_φ is a graduated truth value associated with the formula φ. The pair (φ, μ_φ) is called a *graduated formula*. The deduced truth value $\mu_A \otimes \mu_{A \to B}$ is a lower bound for the real truth value μ_B of the formula *B*. Consequently, the graduated modus ponens can be used to find a lower bound estimate of the truth value of a given formula. Since the deduced truth value need not coincide with the real truth value, we write (φ, μ_φ) instead of $\varphi = \mu_\varphi$.

If a formula φ is the result of repeated application of graduated modus ponens, then a deduced truth value μ_φ of φ can be interpreted as a proof of the fact that φ holds true with a truth degree of at least μ_φ. In RPL different proofs can yield different deduced truth values for the same formula. If the deduced truth value of a formula equals zero, this means that no information on the truth value of the formula can be deduced from the given facts. The largest deduced value that can be derived is the best available estimate for the real truth degree.

6.2 Euclid's First Postulate and Graduated Modus Ponens

Given two extended points *P* and *Q* that are subsets of the extended lines *L* and *M*, we can use the graduated Modus Ponens rule to derive a lower bound for the equality measure *e(L,M)* from the graduated uniqueness property (38).

Before doing the deduction, we simplify formula (38): Since *P* and *Q* are subsets of *L* and *M*, the extended incidence predicate defined in (2) yields the truth value 1 for all instances:

$$on(P,L) = on(Q,L) = on(P,M) = on(Q,M) = 1 . \qquad (44)$$

Since 1 is the unit of the t-norm, $x \otimes 1 = 1$ holds for arbitrary $x \in [0,1]$. Inserting (44) into the graduated uniqueness property (38) yields the following simplified formula[3]

$$\left[\neg e(P,Q)[L] \otimes \neg e(P,Q)[M] \otimes dis(\overline{PQ})\right] \to e(L,M) , \qquad (45)$$

which can be rewritten as

$$dis(\overline{PQ}) \to \underbrace{\left[\neg e(P,Q)[M] \to \underbrace{\left[\neg e(P,Q)[L] \to e(L,M)\right]}_{=:x} \right]}_{=:y} \qquad (46)$$

using the equivalence (11) introduced in chapter 4.2.2.

Now the graduated modus ponens rule can be applied to *e(P,Q)[L]*, *e(P,Q)[M]*, $dis(\overline{PQ})$, and (46). Assume the truth value of *e(P,Q)[L]* is μ, and the truth value of *e(P,Q)[M]* is λ. Both can be calculated from formula (28) using *s(P)*, *s(Q)*, and $\delta(P,Q)$. The truth value η of $dis(\overline{PQ})$ can be estimated from *s(P)* and *s(Q)*. Since (46) is a simplified version of the graduated uniqueness property (38), the truth value of (46) equals 1. A first application of Graduated Modus Ponens to $dis(\overline{PQ})$ and (46) yields

$$\frac{\left(dis(\overline{PQ}), \eta\right), \left(dis(\overline{PQ}) \to y, 1\right)}{\left(y, \eta \otimes 1\right)} , \qquad (47)$$

where $\eta \otimes 1 = \eta$ is the deduced truth value of the formula $y = \left[\neg e(P,Q)[M] \to x\right]$. The next deduction step yields

$$\frac{\left(\neg e(P,Q)[M], \mu\right), \left(\neg e(P,Q)[M] \to x, \eta\right)}{\left(x, \mu \otimes \eta\right)} . \qquad (48)$$

Finally we deduce

$$\frac{\left(\neg e(P,Q)[L],\ \lambda\right),\ \left(\neg e(P,Q)[L] \to e(L,M),\ \mu \otimes \eta\right)}{\left(e(L,M),\ \lambda \otimes \mu \otimes \eta\right)} . \qquad (49)$$

As a result, a lower bound estimate $\lambda \otimes \mu \otimes \eta$ for the *real* truth value ρ of $e(L,M)$ has been derived: $(\lambda \otimes \mu \otimes \eta) \le \rho$. This estimate can be calculated by simple Łukasiewicz conjunction (7). The proposed procedure provides a quick and slim method for finding a lower bound for the degree of equality of two extended lines that are incident with two extended points. The only prior knowledge necessary is the information about the size measures *s(P)*, *s(Q)*, *s(L)*, *s(M)*, and about the distance measure $\delta(P,Q)$.

Given a predefined threshold for the equality of lines, the procedure can be used to produce a warning, if in a construction the value falls below the threshold. The warning means that not enough information is available to derive a non-zero lower bound equality measure. In order to find out, if the equality value indeed equals zero, the exact value of the equality measure must be calculated from the exact knowledge about shapes and relative locations of the involved extended objects. This can be done automatically, without user involvement. Only if the real equality value falls below the threshold must the user be informed that the construction is ill-defined.

The procedure can also be used to estimate how well defined a geometric construction is, if no exact knowledge about shape and location of the involved objects is available. Such a situation may arise in the context of ubiquitous computing, where the ability to represent and query textual descriptions of spatial configurations becomes increasingly important. The lack of detailed information on geographical entities may be caused by a limited bandwidth of hand held devices or by incomplete information from a participatory database.

7. CONCLUSIONS

The paper discusses a special form of positional uncertainty in vector based GIS, namely positional tolerance that arises from geometric constructions with extended primitives. We propose a framework for approximating and propagating positional tolerance through the steps of a geometric construction process. As a first step towards this goal, we address Euclid's first postulate, which lays the foundation for consistent geometric reasoning in all classical geometries. We isolated six qualities that are not present in the Boolean version of Euclid's first calculus, but must be taken into account when introducing extended primitives. We proposed interpretations in the elliptic plane for

[3] The graduated uniqueness property (38) includes the universal quantifier $\inf_{P,Q,L,M}$, which is not needed in the assumption.

the incidence and equality relation for extended primitives, and gave a fuzzification of Euclid's first postulate in Łukasiewicz logic. We provide a proof on the author's website that the proposed interpretations are a valid model of the fuzzified version of Euclid's first postulate. We introduced graduated geometric reasoning with Rational Pavelka logic as a means of approximating and propagating positional tolerance through the steps of a geometric construction.

8. FUTURE WORK

We currently implement the proposed elliptic model in the functional programming language HASKELL. Euclidean data is transformed into homogeneous coordinates. After calculating the equality values in the elliptic model, a local Euclidean approximation is visualized in the $z=1$ plane. After an initial testing phase with artificial data, real GIS data is used to test for practical applicability of the model.

The axiomatic approach to geometry makes it possible to extend the proposed theory to include further geometric primitives and axioms, like e.g. a fuzzifid betweenness relation between extended points, and the corresponding axioms. It is our objective to provide a consistency analysis with the tools of mathematical logic after a number of axioms have been successfully modeled.

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