Projective Spray Can Geometry –Towards an Axiomatic Approach to Error Modelling for Vector Based Geographic Information Systems

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Abstract

The integration of error analysis in geographic information systems (GIS) is a dominating research topic in geographic information science. Current vector-based GIS software is based on idealized geometric objects: Infinitely small points and infinitely thin lines disregard the real character of object representation. The present paper outlines an axiomatic model of 2D projective geometry that incorporates positional random error. As a basis, Menger's axiomatic system for projective geometry is used. Points with errors are modelled by normal distributions and are called Spray Can Points. Lines with errors or Spray Can Lines are defined to be the dual of Spray Can Points. The paper gives proof that these objects fulfill the axioms of projective geometry as proposed by Menger and are therefore suitable as a projective model. In future work the proposed model will be used to formulate a corresponding Euclidean model for objects with errors.

1. Introduction

A fundamental issue of geographic information systems (GIS) is the digital representation and manipulation of spatial objects. The basic geometric components for the representation of objects in vector based GIS are points, lines and polygons. Operations for object manipulation are based on the rules of Euclidean geometry. Geometric functionality in GIS is implemented on the basis of a model of infinitely small points and infinitely thin lines. This is in sharp contrast to the fact that geographic data and their representation are extended and uncertain in location.

The integration of error representation and analysis is often regarded as crucial for the commercial and legal viability of GIS [7]. Users should be able to assess the accuracy of the information upon which they base their decisions [18]. One aspect of error analysis is the assessment of error propagation effects during GIS operations. The propagation of errors may sum up effects and in the worst case leads to meaningless data. Consequently, it should be an integral part of GIS operations.

For example, consider a parcel of land whos corner points are measured with a certain precision. In a GIS, a polygon is constructed from the corner points, representing the boundary of the parcel. Hereby each line segment of the polygon is constructed from two input points by what Menger calls a *join* operation. The line segment can be assigned some positional error measure, depending on the error of the initial corner points of the parcel and the distance betwen them. When computing the intersection of a boundary line with a path traversing the parcel, the resulting intersection point exhibits an error measure that depends on the error of the boundary line, the error of the line representing the path, the intersection angle and the error of the intersection angle. If the intersection point is not measured, but *constructed* in the GIS, the resulting error depends on the geometric construction process.

In the present paper an axiomatic model of 2D projective geometry is defined, that incorporates random error in positional spatial data. Projective geometry provides a more concise axiomatic system than Euclidean geometry. It can be formulated in simple algebraic terms and independently of dimension. It is preferable as a foundation for geometric operations for GIS. Since the Euclidean space can be naturally embedded in the projective space, this approach provides a basis for modeling error propagation in Euclidean geometry.

A *point* in the proposed model is called *Spray Can Point* and is modelled by the two dimensional probability density function (pdf) of a Gaussian normal distribution. Random error in geometric data is usually assumed to be normally distributed [8]. This assumption is based on the central limit theorem stating that the sum of identically independent distributions approximately follows a normal distribution. The name *Spray Can Geometry* is motivated by the way a spray can produces points. The single droplets of paint are randomly distributed over the paper, following a Gaussian distribution. The probability of one droplet falling in the area $dxdy$ is given by the integral over the Gaussian pdf in $dxdy$.

Starting with the definition of a *Spray Can Point*, *Spray Can Lines* are introduced by duality. Based on the axioms of projective geometry proposed by Menger [3] *Spray Can Operations* are defined for the connection of Spray Can Points and the intersection of Spray Can Lines. The paper gives proof that the defined objects and operations fulfill Menger's axioms of projective geometry. The novel contribution is the axiomatic approach to define operations for geometric objects with error.

This paper is divided into 5 sections. Following the introduction, a brief review of literature related to modeling positional random errors in 2D GIS is given. Subsequent to a definition of axiomatic geometry in section 3, the proposed Spray Can model of projective geometry is introduced in section 4. Section 5 concludes the paper and addresses open questions.

2. Modelling positional random error in 2D GIS

The idea for the proposed research emerged from a keynote speech given by Lotfi Zadeh [19] about computing with imprecise data. He suggested an approach replacing the ideal model of extensionless points and lines by points and lines as they are produced by a Spray Can.

Several approaches exist for assessing the positional error of derived geometric objects in vector based GIS. One common way is the simulation method. A Monte Carlo approach can be applied to simulate the probability density function of a line segment based on the pdf of the endpoints. Lei et. al. [12] assume two-dimensional normal distribution for the line endpoints. Random line segments are generated using a stochastic generator. An error model for the line segment is developed for independent and dependent endpoints. Abbaspour et. al. [1] use Monte Carlo simulation for testing the error propagation behavior of overlay operations for polygons.

Heuvelink et. al. [10] present a comprehensive probabilistic framework for specifying, representing and simulating uncertain environmental variables, including positional and attribute uncertainty. The positional uncertainty behavior of a geometrical object is specified by assigning appropriate pdfs to primitive points of the object (e.g. the corners of a poygon) or to reference points of the object (e.g. the objects centroid). Realizations of the modelled behavior can be used as input for Monte Carlo uncertainty propagation studies. In contrast to this, the present paper gives an axiomatic model of geometry that incorporates error propagation as an integral part.

Another way of modeling positional error of geometric objects are buffer based models. For a point with error a buffer can be defined by a confidence region containing the true location of the point with a probability larger than a predefined confidence level [16]. If the accuracy and dependencies of the coordinates is known, the error of a point can be represented by the standard error ellipse [8].

For line segments the concept of a buffer based model goes back to Perkal's definition of the epsilon band model (cited in [13]). An epsilon band defines a buffer around a line segment containing the true line with some probability. Many variants of the epsilon band model have been developed since then. Alesheikh and Li [2] represent the error of a line segment by the union of error ellipses of all points along that line depending on the positional errors of the two endpoints. Shi [16] introduced the error band model with a rectangular confidence region of non-uniform width. He considers the endpoints of the line segment to be statistically independent. This model was generalized by Shi and Liu [17] for statistically dependent endpoints, named the G-band model.

Leung et. al. [14] introduced the covariance-based error band model for line segments that contains the classical epsilon band model and the error band model as special cases. Assuming a multivarate normal distribution for points with errors, lines with errors are derived by strictly applying the approximate law of error propagation. Formulas are derived for the intersection of lines with errors, resulting again in points with errors. Closedness of operations is therefore achieved. The present approach likewise addresses the issue of closedness of operations, but differs from [14] by aiming at a closed axiomatic geometry for objects under positional random error. An axiomatic approach has the advantage that it produces consistent solutions for derived operations and relations.

Clementini [5] specifies a model of uncertain lines as an extension of the model for regions with a broad boundary. Lines with a broad boundary incorporate all types of uncertainties occuring in linear spatial objects and can be easily integrated into existing data models for spatial databases. In constrast to the present work, the model aims at the study of topological relations between uncertain lines.

In his thesis Heuel [9] proposes a method for statistical reasoning for polyhedral object reconstruction. He proposes a framework of uncertain projective geometry using Grassman-Cayley algebra. His work is similar to the approach presented in this paper, but focuses at photogrammetric applications. The formalism used builds on algebraic invariants of projective geometry but is not a direct model of an axiomatic system.

The research proposed here aims at making use of the power of a closed and fully defined axiomatic system. An error model that complies with the axioms of projective geometry automatically provides consistency, i.e. it avoids contradictions. Equipped with a realistic interpretation of geometric primitives, all geometric objects can be derived.The standard operations of vector based GIS software are built upon the axioms of Euclidean geometry. An error model complying with these axioms allows to reuse the analysis functions already defined. The concept of lifting (cf. [15]) provides an appropriate tool for the extension of exact functions to Spray Can functions. The novel contribution of this work is the use of axiomatic geometry to build an error model of objects under positional random error.

3. Axiomatic Geometry

Euclid's *Elements* is often referred to as the most successful textbook in the history of mathematics, since it introduced the deductive method to formal sciences. Its aspiration was to logically deduce all theorems of geometry from few obvious statements and rules that need not be justified. These statements are nowadays called *undefined terms* or *geometric primitives*; the rules are called *postulates* or *axioms*.

A *model* of Euclidean geometry is an interpretation of primitives that fulfills the axioms. A metaphor commonly attributed to David Hilbert states that it should always be possible to replace the notions *point*, *line* and *plane* by *desk*, *bank* and *beer mug*, as long as the rules apply. The modern and fully consistent axiomatic approach determines the classical Euclidean geometry up to isomorphism.

The principle of coexisting isomorphic models can be employed for all modern geometric systems, including projective geometry. It is used in the present paper to define a *Spray Can model* of projective geometry. *Spray Can objects* are interpretations of primitive objects incorporating positional random error. *Spray Can relations* are interpretations of primitive relations that operate on Spray Can objects. They must be defined accordingly, so that the axioms apply.

4. A spray can model for projective geometry

The Euclidean plane is naturally embedded in the projective plane. The *extended Euclidean plane* can be constructed by adding to the Euclidean plane an ideal "line at infinity", the horizon. The Line at infinity consists of distinct points called *points at infinity*. For a given line l in the real plane $\mathbb{R}^2$, all lines parallel to l are said to intersect at a point at infinity, which is uniquely determined by the direction of l [4]. The extended Euclidean plane is isomorphic to the projective plane.

Projective geometry uses the powerful concept of duality. Projective duality allows to exchange the notions of *point* with *line* and *connect (join)* with *intersect (meet)* and vice versa without violating the validity of a theorem. As a consequence, an axiomatic system of projective geometry requires only half of the axioms, provided a duality operation has been defined. Harold Coxeter ([6], p.231) describes the principle of duality to be "one of the most elegant properties of projective geometry".

It was the goal of many mathematicians to find an axiomatic system for Euclidean and projective geometry that is as simple and concise as possible. One of them was Karl Menger, son of the famous economist Carl Menger. Karl Menger was born in Vienna, but immigrated to USA during the second world war. In the work *Studies in Geometry* [3] Menger gave an axiomatization of projective geometry consisting of 5 primitives and 6 axioms, 3 of which are simply dual statements of the others. This is a big difference to Hilbert's axiomatic system for Euclidean plane geometry, consisting of 6 primitives and 14 axioms.

Menger's axiomatization is a purely algebraic approach which, as a consequence, is dimension independent. The constructions in Hilbert's approach depend on the dimension of the space under consideration [11].

4.1. Menger's Axiomatization of Projective Geometry

The primitive objects of Menger's axiomatic system are a set S of objects of concern and two special objects contained in S. These special objects are *vacuum* V, representing *nothing*, and *universe* U, representing *everything*. The primitive operations in Menger's definition are *join* and *meet*. *Join*, denoted by $\vee$, connects two elements of S, *meet*, denoted by $\wedge$, intersects two elements of S.

The role of *vacuum* and *universe* is axiomatized by four postulates:

1.1) $U \vee X = U, \qquad U \wedge X = X$

1.2) $V \wedge X = V, \qquad V \vee X = X$

for all X in S.

The behavior of arbitrary projective objects is determined by two *projective laws*:

2.1) $X \vee ((X \vee Y) \wedge Z) = X \vee ((X \vee Z) \wedge Y)$

2.2) $X \wedge ((X \wedge Y) \vee Z) = X \wedge ((X \wedge Z) \vee Y)$.

The axioms 1.1 and 1.2 are dual to each other. 1.2 is obtained from 1.1 by replacing each occurrence of U by V and each occurrence of $\vee$ by $\wedge$ and vice versa. The same holds for the axioms 2.1 and 2.2.

Menger's axiomatization is a dimension-independent definition of projective geometry. In two-dimensional projective geometry, the set S of objects of concern is called the *projective plane* $\mathbb{P}^2$. It resolves into four distinct groups: U, V, points and lines [3]. U is dual to V and points are dual to lines. The *duality principle of projective plane geometry* states, that a valid theorem remains true, if the following terms are interchanged:

$$point \leftrightarrow line$$
$$V \leftrightarrow U$$
$$join \leftrightarrow meet.$$

To define a Spray Can model of projective geometry, it is sufficient to define instances of the primitives *point*, U and *join* and a *duality* operation for them.

4.2. The spherical model of the projective plane

A common way to model the two-dimensional projective plane $\mathbb{P}^2$ is the unit sphere $S^2 \subset \mathbb{R}^3$ where antipodal points are identified (cf. Figure 1). In this model projective points are represented by antipodal pairs of $\mathbb{R}^3$-points on the sphere and projective lines are represented by great circles. As a consequence, three-dimensional real coordinates can be assigned to projective points and lines.

More precisely, a projective point can be represented by a pair of $\mathbb{R}^3$-points $(p, -p)$, where p has unit length. Each pair of points uniquely determines a direction in $\mathbb{R}^3$.

A projective line $l \in \mathbb{P}^2$ can be represented by a great circle in S^2. Each great circle uniquely determines a plane through the origin of $\mathbb{R}^3$. The equation of the plane, $Ax + By + Cz = 0$, is again uniquely determined by the triple $(A, B, C) \in \mathbb{R}^3$, where (A, B, C) is the vector normal to the plane. Consequently, a great circle can be represented by the pair $((A, B, C), -(A, B, C))$ of antipodal points perpendicular to it.

Projective points and lines are *dual* to each other. Each projective point $(p, -p) = ((A, B, C), -(A, B, C))$ can be uniquely mapped to the projective line $l = d\,((p, -p)) = \{(x, y, z) \in \mathbb{R}^3 \ : \ Ax + By + Cz = 0\}$ in the dual space. The duality operation $d : \mathbb{P}^2 \to d(\mathbb{P}^2)$ can be visualized by the bijection mapping each point p on the sphere to the great circle perpendicular to p (cf. Figure 1). For the operation d idempotency holds [4], i.e.

$$d.d = id, \tag{1}$$

where . denotes the composition of functions.

Two projective points $(q, -q), (r, -r) \in \mathbb{P}^2$ uniquely determine a projective line

$$l = (q, -q) \vee (r, -r) = d\big((q \times r), -(q \times r)\big)\ , \tag{2}$$

where $\vee$ denotes the *join* operation and $\big((q{\times}r), -(q{\times}r)\big) \in d(\mathbb{P}^2)$ is a projective point in the dual space of $\mathbb{P}^2$ (cf. Figure 2a).

In the following, the unit sphere modulo antipodes will be denoted by $S^2/\pm$. It is isomorphic to the projective plane: $(S^2/\pm) \cong \mathbb{P}^2$ [4].

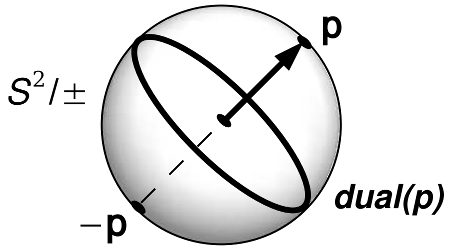

Figure 1. Duality in the unit sphere model of the projective plane.

4.3. Spray Can Points

The pdf of the two-dimensional *Gaussian normal distribution* $G(\mu, \Sigma)(x) = \mathbb{R}^2 \to \mathbb{R}^2$ is defined on the Euclidean plane $\mathbb{R}^2$. G is uniquely determined by its mean μ and its covariance matrix Σ:

$$G_{(\mu,\Sigma)}(x) = \frac{1}{2\pi \cdot \sqrt{|\Sigma|}} \cdot e^{\left(-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right)}.$$

For a point $p \in S^2 \subset \mathbb{R}^3$ let $B_p = \{b_1, b_2, p\}$ be an orthonormal basis of $\mathbb{R}^3$ and $B_p^{'} = \{b_1, b_2\}$ its two-dimensional restriction to the orthogonal subspace perpendicular to p. The transformation of a point $x = [x_1, x_2]_{B_p'} \in \mathbb{R}^2$ to its homogeneous coordinates (with respect to B_p) is given by

$$h_{B_p} : \mathbb{R}^2 \to H_p \subset \mathbb{R}^3, \quad [x_1, x_2]_{B_p'} \mapsto [x_1, x_2, 1]_{B_p},$$

where H_p is the homogeneous plane with respect to B_p,

$$H_p \quad = \quad \left(\{x = [x_1, x_2, x_3]_{B_p} \ : \ x_3 \neq 0\}/\equiv\right) \quad \text{where}$$
$$x \equiv \bar{x} \quad \Leftrightarrow \quad x = k \cdot \bar{x}, \quad k \in \mathbb{R}.$$

As indicated in Figure 2b, the function h_{B_p} is an isomorphism, i.e. $H_p \cong \mathbb{R}^2$. The inverse transformation is given by

$$h_{B_p}^{-1} : H_p \to \mathbb{R}^2, \quad [x_1, x_2, x_3]_{B_p} \mapsto [x_1/x_3, x_2/x_3]_{B_p'}.$$

For $\mathcal{O} = (0, 0) \in \mathbb{R}^2$ we have $h_{B_p}(\mathcal{O}) = [0, 0, 1]_{B_p} = p$ and hence $h_{B_p}^{-1}(p) = \mathcal{O}$.

Definition 1 (Spray Can Point) For a projective point $(p, -p) \in (S^2/\pm)$ and a covariance matrix $\Sigma \subset \mathbb{R}^4$ a *Spray Can Point (SCP)* $\tilde{p}_\Sigma$ on the projective plane is a function $\tilde{p}_\Sigma : \mathbb{P}^2 \cong (S^2/\pm) \to \mathbb{R}$,

$$\tilde{p}_\Sigma\left((x, -x)\right) = \begin{cases} G_{(\mathcal{O},\Sigma)} \,.\, h_{B_p}^{-1}(x) & \text{if } (x, -x) \in H_p \\ 0 & \text{else.} \end{cases}$$

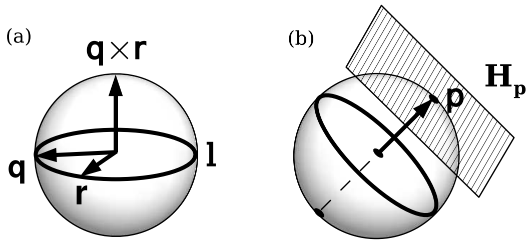

Figure 2. (a) Two projective points q, r determine a projective line $\mathrm{l} = \mathrm{dual}(\mathrm{q} \times \mathrm{r})$. (b) The homogeneous plane $\mathrm{H_p}$.

$G_{(\mathcal{O},\Sigma)}$ is the pdf of a Gaussian normal distribution on $H_p \cong \mathbb{R}^2$ with mean $\mu = \mathcal{O} = (0,0) \in \mathbb{R}^2$ and covariance matrix Σ. A SCP assigns every projective point $(x, -x)$ the corresponding value of the Gaussian pdf (cf. Figure 3a).The point $(p, -p)$ is assigned the maximum value. □

Definition 2 (Spray Can Universe) The *Spray Can Universe* $\tilde{U}$ is defined to be the union of all Spray Can Points: $\tilde{U} = \{\tilde{p}_\Sigma : H_p \to \mathbb{R} \ : \ p \in \mathbb{P}^2, \Sigma \in \mathbb{R}^4\}$. □

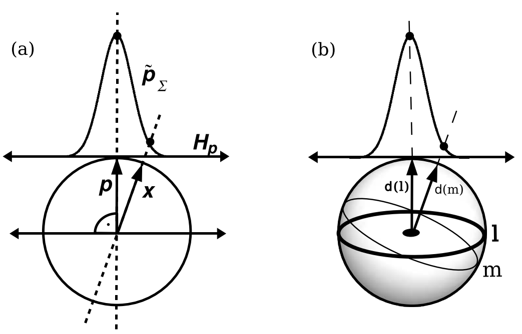

Figure 3. (a)A Spray Can Point $\tilde{\mathrm{p}}_\Sigma$ in p with covariance matrix Σ assigns a value to every projective point x. (b) A Spray Can Line $\tilde{\mathrm{l}}_\Sigma$ in l assigns a value to every projective line m.

4.4. Spray Can Duality

A Spray Can version of duality can be defined by utilizing the duality operation *d* of the exact spherical model of the projective plane with infinitely small points and infinitely thin lines (cf. Figure 1). The exact dual operation operates on exact projective objects $X, Y, Z, \ldots$. Likewise, the *Spray Can dual* operates on Spray Can Objects $\tilde{X}_{\Sigma_X}, \tilde{Y}_{\Sigma_Y}, \tilde{Z}_{\Sigma_Z} \ldots$. A *Spray Can Object (SCO)* in $X \in \mathbb{P}^2$ is a map $\tilde{X}_{\Sigma_X} : \mathbb{P}^2 \to \mathbb{R}$, depending on an exact projective object X and a covariance matrix $\Sigma \in \mathbb{R}^4$.

Definition 3 (Spray Can Dual) For a Spray Can Object $\tilde{X}_\Sigma : \mathbb{P}^2 \to \mathbb{R}$ in $X \in \mathbb{P}^2$ let the *Spray Can Dual (scd)* of $\tilde{X}_\Sigma$ be a Spray Can Object in $d(X) \in d(\mathbb{P}^2)$, defined by $scd.\tilde{X}_\Sigma : d(\mathbb{P}^2) \to \mathbb{R}$ with

$$scd.\tilde{X}_\Sigma = \tilde{X}_\Sigma.d,$$

where . denotes the concatenation of functions. The Spray Can Dual of the Spray Can Universe is defined to be the empty set: $scd(\tilde{U}) = \emptyset$. □

From idempotency of *d* (cf. equation 1) idempotency of *scd* follows:

$$scd.scd.\tilde{X}_\Sigma = \tilde{X}_\Sigma.d.d = \tilde{X}_\Sigma.id = \tilde{X}_\Sigma.$$

4.5. The Spray Can Join Operation

A Spray Can version for the *join* of two Spray Can objects can be defined by using the join operator $\vee$ of the exact spherical model (cf. Chapter 4.1). In the following we will use the symbol $\vee$ for both models.

Definition 4 (Spray Can Join) We define a *Spray Can Join (scjoin)* operation $\vee$ for the connection of two Spray Can Objects $\tilde{X}_{\Sigma_X}$ and $\tilde{Y}_{\Sigma_Y}$ by $(\tilde{X}_{\Sigma_X} \vee \tilde{Y}_{\Sigma_Y}) : \mathbb{P}^2 \to \mathbb{R}$,

$$\tilde{X}_{\Sigma_X} \vee \tilde{Y}_{\Sigma_Y} = scd.\big[d(X \vee Y)\big]^{\sim}_{\Sigma_{X \vee Y}},$$

where X, Y are exact projective objects and $X \vee Y$ denotes the join of X and Y in the exact model. $[d(X \vee Y)]^{\sim}_{\Sigma_{X \vee Y}}$ is a Spray Can Object in $d(X \vee Y)$ in the dual space. The SCJoin of a Spray Can Object with the Spray Can Universe is the Spray Can Universe: $\tilde{X}_{\Sigma_X} \vee \tilde{U} = \tilde{U}$. □

The SCJoin operation for Spray Can Points $\tilde{p}_{\Sigma_p}$ and $\tilde{q}_{\Sigma_q}$ is well defined:

$$\begin{aligned} \tilde{p}_{\Sigma_p} \vee \tilde{q}_{\Sigma_q} &= scd.\Big[d\Big((p,-p) \vee (q,-q)\Big)\Big]^{\sim}_{\Sigma_{p \vee q}} \\ &= scd.\big[p \times q\big]^{\sim}_{\Sigma_{p \times q}}. \end{aligned} \quad (3)$$

$[p \times q]^{\sim}_{\Sigma_{p \times q}}$ is a SCP in $(p \times q) \in d(\mathbb{P}^2)$ in the dual space (cf. equation 2). Note that $\Sigma_{p \times q}$ can be obtained by applying the approximate law of error propagation to the function $cross : \mathbb{R}^3 \to \mathbb{R}^3 \to \mathbb{R}^3$, $cross(x, y) = x \times y$ [8].

4.6. Construction of Dual Primitives

Provided with the definitions of $\tilde{U}$, *SCP* and *SCJoin* the concepts of *Spray Can Line*, *Spray Can Meet* and *Spray Can Vacuum* follow by duality.

Definition 5 (Spray Can Line) For a projective line $l \in \mathbb{P}^2$ with $d(l) = p$ and a covariance matrix $\Sigma \in \mathbb{R}^4$ a *Spray Can Line (SCL)* is a function $\tilde{l}_\Sigma : \mathbb{P}^2 \to \mathbb{R}$ with

$$\tilde{l}_\Sigma = scd.\tilde{p}_\Sigma \ ,$$

where $\tilde{p}_\Sigma$ is a SCP in the dual space. A SCL $\tilde{l}_\Sigma$ assigns every projective line $m \in \mathbb{P}^2$ the value $\tilde{l}_\Sigma(m) = \tilde{p}(d(m))$ of its dual point $d(m) \in d(\mathbb{P}^2)$ (cf. Figure 3b). □

The Spray Can Dual of a SCL $\tilde{l}_\Sigma : \mathbb{P}^2 \to \mathbb{R}$ is a SCP:

$$\begin{aligned} &scd.\tilde{l}_\Sigma : \mathbb{P}^2 \to \mathbb{R}, \\ &scd.\tilde{l}_\Sigma = scd.scd.\tilde{p}_\Sigma = \tilde{p}_\Sigma. \end{aligned}$$

The result of the *scjoin* operation introduced in chapter 4.4 of two Spray Can Points is a Spay Can Line (cf. equation (3)).

Definition 6 (Spray Can Meet) We define a *Spray Can Meet (scmeet)* operation $\wedge$ for the intersection of two Spray Can Objects $\tilde{X}_{\Sigma_X}$ and $\tilde{Y}_{\Sigma_Y}$ by $(\tilde{X}_{\Sigma_X} \wedge \tilde{Y}_{\Sigma_Y}) : \mathbb{P}^2 \to \mathbb{R}$,

$$\tilde{X}_{\Sigma_X} \wedge \tilde{Y}_{\Sigma_Y} = scd(\, scd.\tilde{X}_{\Sigma_X} \vee scd.\tilde{Y}_{\Sigma_Y}\,).$$

Definition 7 (Spray Can Vacuum) The Spray Can Vacuum $\tilde{V}$ is the Spray Can Dual of the Spray Can Universe, $\tilde{V} = scd(\tilde{U}) = \emptyset$. □

Spray Can duality maps Spray Can Objects in the primal space $\mathbb{P}^2$ to Spray Can Objects in the dual space $d(\mathbb{P}^2)$ in the following way:

$$\begin{aligned} SCP &\leftrightarrow SCL \\ scjoin &\leftrightarrow scmeet \\ \tilde{U} &\leftrightarrow \tilde{V} \end{aligned} \tag{4}$$

4.7. Verification of Axioms

In this subsection Menger's axioms of projective geometry (cf. chapter 4.1) are verified for the objects and operations of the Spray Can Model introduced in the foregoing chapters. As main tools the duality principle (4) for Spray Can Objects and Menger's axioms for exact projective objects are applied.

PROOF (AXIOMS 1.1, 1.2) The first axiom of axiom set (1.1) holds by definition 4: $\tilde{U} \vee \tilde{X}_\Sigma = \tilde{U}$. The second axiom in (1.1), $\tilde{U} \wedge \tilde{X}_\Sigma = \tilde{X}$, follows by applying definitions 4 and 6:

$$\begin{aligned} \tilde{U} \wedge \tilde{X}_\Sigma &= scd.\left(scd.\tilde{U} \ \vee \ scd.\tilde{X}_\Sigma\right) \\ &= scd.scd.\big[\, d\,(\, V \vee d(X)\,)\,\big]^\sim_{\Sigma_{(V \vee d(X))}} \\ &= \big[\, d.d(X)\,\big]^\sim_{\Sigma_{d(X)}} = \tilde{X}_\Sigma \end{aligned}$$

Here the last line follows from $V \vee d(X_\Sigma) = d(X_\Sigma)$ and $\Sigma_{d(X)} = \Sigma_X = \Sigma$.

The first axiom in axiom set (1.2), $\tilde{V} \wedge \tilde{X}_\Sigma = \tilde{V}$, follows from axiom set (1.1) by Spray Can duality. Interchanging the terms (4) in $\tilde{U} \vee \tilde{X}_\Sigma = \tilde{U}$ yields $\tilde{V} \wedge \tilde{X}_\Sigma = \tilde{V}$. Alternatively this can be verified by applying definition 7 and idempotency of *scd*:

$$\tilde{V} \wedge \tilde{X}_\Sigma = scd\left(scd.\tilde{V} \vee scd.\tilde{X}_\Sigma\right) = scd(\tilde{U}) = \tilde{V}.$$

Likewise, interchanging the terms (4) in $\tilde{U} \wedge \tilde{X}_\Sigma = \tilde{X}$ yields $\tilde{V} \vee \tilde{X}_\Sigma = \tilde{X}$. ■

PROOF (AXIOMS 2.1, 2.2) To verify axiom (2.1), note that

$$(\tilde{X}_{\Sigma_X} \vee \tilde{Y}_{\Sigma_Y}) \wedge \tilde{Z}_{\Sigma_Z} = \big[(X \vee Y) \wedge Z\big]^\sim_{\Sigma_{(X \vee Y) \wedge Z}} \tag{5}$$

holds, following from definitions 4 and 6:

$$\begin{aligned} &(\tilde{X}_{\Sigma_X} \vee \tilde{Y}_{\Sigma_Y}) \wedge \tilde{Z}_{\Sigma_Z} = \\ &\quad = \left(\ scd.\big[d(X \vee Y)\big]^\sim_{\Sigma_{X \vee Y}}\right) \ \wedge \ \tilde{Z}_{\Sigma_Z} \\ &\quad = scd.\left(scd.scd.\big[d(X \vee Y)\big]^\sim_{\Sigma_{X \vee Y}} \ \vee \ scd.\tilde{Z}_{\Sigma_Z}\right) \\ &\quad = scd.scd.\left[d\Big(d(X \vee Y) \ \vee \ d(Z)\Big)\right]^\sim_{\Sigma_{d(X \vee Y) \vee d(Z)}} \\ &\quad = \big[(X \vee Y) \ \wedge \ Z\big]^\sim_{\Sigma_{(X \vee Y) \wedge Z}} . \end{aligned}$$

Consequently for the left side of axiom (2.1) we find

$$\begin{aligned} &\tilde{X}_{\Sigma_X} \vee \left((\tilde{X}_{\Sigma_X} \vee \tilde{Y}_{\Sigma_Y}) \wedge \tilde{Z}_{\Sigma_Z}\right) = \\ &\quad = \tilde{X}_{\Sigma_X} \vee \big[(X \vee Y) \ \wedge \ Z\big]^\sim_{\Sigma_{(X \vee Y) \wedge Z}} \\ &\quad = scd.\left[d\Big(X \ \vee \ (\ (X \vee Y) \wedge Z\ )\Big)\right]^\sim_{\Sigma_{(X \vee Y) \wedge Z}} \\ &\quad \underset{(*)}{=} scd.\left[d\Big(X \ \vee \ (\ (X \vee Z) \wedge Y\ )\Big)\right]^\sim_{\Sigma_{(X \vee Z) \wedge Y}} \\ &\quad = \tilde{X}_{\Sigma_X} \vee \left((\tilde{X}_{\Sigma_X} \vee \tilde{Z}_{\Sigma_Z}) \wedge \tilde{Y}_{\Sigma_Y}\right) . \end{aligned}$$

In step $(*)$ axiom (2.1) for exact projective objects is applied. This proves axiom (2.1) for Spray Can Objects. Axiom (2.2) can be obtained by interchanging the terms (4) in axiom (2.1). ■

5. Conclusions and further work

We have shown that an axiomatization of a realistic treatment of projective geometry can be achieved. The long term objective of this research is to define a pertinent realistic model of Euclidean geometry for GIS. Current GIS are based on an idealized axiomatization of geometry which deals with points and lines without extensions. Real points and lines have extensions and the locations of idealizations are not precisely measurable. Lotfi Zadeh has suggested the geometry produced when drawing with a spray can as an inspiration for a realistic model of geometry.

A spray can produce a point with a random distribution of color droplets, approximating a Gaussian normal distribution. The paper drafts how to construct a consistent axiomatic geometry with this model.

For an axiomatization we use Mengers axioms of projective geometry, which are fewer and simpler than Hilberts axioms for a Euclidean geometry. The duality between points and lines in projective geometry further simplifies the task. It is necessary to show how to embed a Spray Can Point into the projective plane and then define two of the three operations *intersection*, *connection* and *duality*. We have given definitions for *duality* and *connection*. *Intersection* follows by duality. We proved that the proposed Spray Can Model of projective geometry complies with Menger's axioms.

The used axiomatic approach assures consistency for the treatment of all geometric operations derived from this foundation. The advantage of these definitions over other more pragmatic efforts in the past is that the implementation of the basic operations is sufficient to extend derived operations, analytical functions and tests for relations to treat geometry with positional uncertainty without danger of contradiction.

In a next step the *join* and *meet* operations introduced in this paper will be extended to allow a dimension independent formulation of the model. 2D and 3D operations will be formulated as special cases. The definitions introduced must be extended and modified to define a model of Euclidean geometry with uncertainty in the location. This model will allow the integration of error analysis and error propagation in current GIS software by accessing standard operations of vector-based GIS.

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