Qualitative Spatial Reasoning: Cardinal Directions as an Example[1]

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Abstract

Geographers use spatial reasoning extensively in large-scale spaces, i.e., spaces that cannot be seen or understood from a single point of view. Spatial reasoning differentiates several spatial relations, e.g. topological or metric relations, and is typically formalized using a Cartesian coordinate system and vector algebra. This quantitative processing of information is clearly different from the ways humans draw conclusions about spatial relations. Formalized qualitative reasoning processes are shown to be a necessary part of Spatial Expert Systems and Geographic Information Systems.

Addressing a subset of the total problem, namely reasoning with cardinal directions, a completely qualitative method, without recourse to analytical procedures, is introduced and a method for its formal comparison with quantitative formulae is defined. The focus is on the analysis of cardinal directions and their properties. An algebraic method is used to formalize the meaning of directions. The standard directional symbols (N, W, etc.) are supplemented with a symbol corresponding to an undetermined direction between points too close to each other which greatly increases the power of the inference rules. Two specific systems to determine and reason with cardinal directions are discussed in some detail.

From this example and some other previous work, a comprehensive set of research steps is laid out, following a mathematically based taxonomy. It includes the extension of distance and direction reasoning to extended objects and the definitions of other metric relations that characterize situations when objects are not disjointed. The conclusions compare such an approach with other concepts.

1. Introduction

Qualitative spatial reasoning is widely used by humans to understand, analyze, and draw conclusions about the spatial environment when the information available is in qualitative form, as in the case of text documents.

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Tobler and Wineberg (1971), for example, tried to reconstruct spatial locations of historic places from scant descriptions in old documents. Verbal information about locations of places can leave certain aspects imprecise and humans deduce information from such descriptions (for example in order to analyze a description of a location in natural science collections (McGranaghan 1991)). The use of cardinal directions is typical for reasoning in geographic or large-scale space, but other spatial relations are equally important.

Most formalizations or implementations of spatial reasoning rely on the Euclidean geometry and the Cartesian coordinate system. There is a clear need for a fully qualitative system of spatial reasoning, combining topological and metric relations. In qualitative reasoning, a situation is characterized by variables which '*can only take a small, predetermined number of values*' (de Kleer and Brown 1985, p. 116) and the inference rules use these values and not numerical quantities approximating them. It is clear that the qualitative approach loses some precision, but simplifies reasoning and allows deductions when precise information is not available.

The work presented here is part of a larger effort to understand how space and spatial situations are described and explained. Within the research initiative 2, 'Languages of Spatial Relations' of the National Center for Geographic Information and Analysis (NCGIA 1989) a need for multiple formal descriptions of spatial reasoning—both quantitative-analytical and qualitative—became evident (Mark, et al. 1989a), (Mark and Frank 1990), (Frank and Mark 1991), (Frank 1990b), (Frank 1990a).

Three special arguments for research in qualitative spatial reasoning serve as examples for numerous others:

- the formalization required for the implementation in Geographic Information Systems (GIS), or generally for spatial information systems,
- the interpretation of spatial relations expressed in natural language, and
- the comparison of the semantics of spatial terms in different languages.

Formalizing spatial reasoning is a precondition for the implementation of any spatial data processing method. Most methods currently used for spatial reasoning translate the problem from the quantitative to the qualitative realm and use analytical geometry for the solution (Dutta 1990, Webster 1990). This is not always an appropriate solution. The treatment of the uncertainty inherent in qualitative spatial descriptions causes problems that cannot be overcome easily in methods translating the problem to the quantitative realm (McDermott and Davis 1984). For this reason, the construction of expert systems which deal with space and spatial problems has been recognized as being difficult (Bobrow, et al. 1986). Abstract, non-coordinate-based methods are necessary for the planning of the execution of spatial queries and their optimization. This is most urgent for intelligent spatial query languages for Geographic Information Systems (Egenhofer 1991). Specifically, algebraic properties of the operations, primarily commutativity and associativity, determine the optimization steps possible.

It has been suggested that user interfaces currently available in Geographic Information Systems are strongly influenced by Anglo-saxon concepts which are difficult to translate to users from a different cultural or linguistic background (Campari 1991), (Mark, et al. 1989b). This problem will increase in importance, as GIS become widely used outside of the Anglo-saxon world and may in the long run limit their usefulness to address the pressing needs of resource management and spatial planning in third world countries. A necessary, but not sufficient step to address this problem is to identify and formalize the concepts used in spatial reasoning, so one can compare methods used in one language with concepts present in another one (Campari and Frank 1994), (Campari and Frank 1993).

The structure of this paper is as follows: The next section describes large-scale space and discusses in some detail reasoning with cardinal directions as an example of spatial inference in a rule based system. Section 3 outlines desirable properties of reasoning with cardinal direction from geometric intuition. Section 4 describes alternative approaches to reason about cardinal directions based on the concepts of angular directions and projections. Section 5 outlines previous approaches to spatial reasoning and links the different perspectives. Section 6 lays out a comprehensive set of research steps which extends distance and direction reasoning to arbitrary objects and shows how to search for additional metric relations that characterize the situations when objects are not disjoint. The conclusion summarizes the results and briefly discusses alternative guiding principles for research in spatial reasoning.

2. Reasoning about Cardinal Directions in Geographic Space

Spatial reasoning is commonplace and defines the conceptualization of a situation as spatial. The details of spatial reasoning, the specific inference rules, depend on the type of space to which it applies. Conceptualization of space does not always lead to the customary Euclidean geometry, but depends on the situation. Zubin differentiates four different spaces, covering the range from the small objects in an area which can be seen at a single glance, to the large areas, where one accumulates knowledge about the space by moving through it (Mark, et al. 1989a) (Couclelis 1992), (Montello 1993). Similar differentiations have been made previously; for example, Kuipers and Levit (1990, p. 208) define large-scale space as

> "a space whose structure is at a significantly larger scale than the observations available at an instant. Thus, to learn the large-scale structure of the space, the traveler must necessarily build a cognitive map of the environment by integrating observations over extended periods of time, inferring spatial structure from perceptions and the effects of actions".

Geographic Information Systems, and geography and cartography in general, deal with this large-scale space, and this justifies the focus of the paper. Moreover, cardinal directions presented as an extensive example, are almost exclusively used in large-scale space. Reports of statements like "the pencils are in the upper, west drawer" - which is said to be used in some mid-west families - or a description as

> CHRISTY: "He gave a drive with the scythe, and I gave a leap to the east. Then I turned around with my back to the north, ... " (Synge 1941, The Playboy of the Western World, p. 132)

which apply cardinal directions to smaller spaces, seems strange or quaint to most.

Cartographic rendering of large-scale spaces in reduced scale transfers the situation to small-scale space and makes concepts from small-scale spatial reasoning available for the analysis of geographic space. An extreme example of applying small-scale space concepts to large-scale space are maps that compare the area of two states by superimposing one onto the other, moving states in space as one would move books on a table. One can, therefore, not limit investigations to a single space and the spatial relations valid in it, but consider the transfer of concepts of spatial reasoning from one to another.

A standard approach to spatial reasoning is to translate the problem into analytical geometry and to use quantitative methods for its solution. Many problems can be conveniently reformulated as optimization with a set of constraints, e.g. location of a resource and shortest path. A similar approach of mapping a situation to coordinate space, has been applied to understand spatial reference in natural language text (Herskovits 1986, Nirenburg and Raskin 1987, Retz-Schmidt 1987). A special problem is posed by the inherent uncertainties in these descriptions and their translation into an analytical format. McDermott (1984) introduced a method using 'fuzz' and in (Dutta 1988, Dutta 1990, Retz-Schmidt 1987) fuzzy logic (Zadeh 1974) is used to combine such approximately metric data.

A qualitative approach to spatial reasoning does not rely on a coordinate plane and does not attempt to map all information into this framework. It can deal with imprecise data and, therefore, yields less precise results than the quantitative approach (Freska 1991). This is highly desirable (Kuipers 1983, NCGIA 1989), because

- precision is not always desirable and
- precise, quantitative data is not always available.

Verbal descriptions have the advantage, that they need not be metrically precise, just sufficient for the task intended. Imprecise descriptions are necessary in query languages where one specifies some property that the requested data should have. Consider, for example, the query "find all factories about 3 miles southeast of town A and northwest of town B. The depiction in Figure 1 is oversimplified and inaccurate, while the visualization of Figure 2 is too complex to show the desired information only.

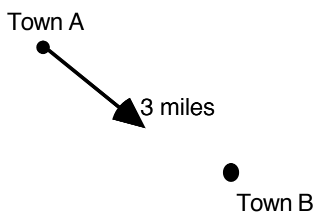

Figure 1: Overspecific visualization

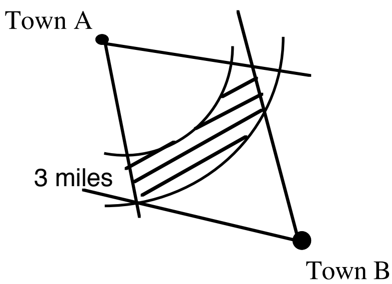

Figure 2: Complex visualization

In large-scale space several spatial relations are important. Pullar and Egenhofer (1988) classified spatial relations in:

- *direction* relations that describe order in space (e.g. *north*, *northeast*),
- *topological* relations that describe neighborhood and incidence (e.g. *disjoint*),
- *comparative* or *ordinal* relations that describe inclusion or preference (e.g. *in*, *at*),
- *distance* relations such as *far* and *near* and
- *fuzzy* relations such as *next to* and *close*.

This paper concentrates on reasoning with cardinal directions as a major example. The specific problem addressed in this example is the following: Given the facts that San Francisco is west of St.Louis and Baltimore is east of St. Louis, one can deduce the direction from San Francisco to Baltimore by the following chain of deductions:

1. Use 'San Francisco is west of St. Louis' and 'Baltimore is east of St. Louis', to establish a sequence of directions San Francisco - St. Louis - Baltimore .

2. Deduce 'St. Louis is west of Baltimore' from 'Baltimore is east of St. Louis'
3. Use the concept of transitivity: 'San Francisco is west of St. Louis' and 'St. Louis is west of Baltimore', thus conclude 'San Francisco is west of Baltimore' .

This paper formalizes such rules and makes them available for inclusion in an expert system.

One is tempted to apply first-order predicate calculus to spatial reasoning with directions:

> "The direction relation NORTH. From the transitive property of NORTH one can conclude that if A is NORTH of B and B is NORTH of C then A must be NORTH of C as well (Mark, et al. 1989a)"

The description of directional relationships between points in the plane can be formulated as propositions 'A is north of B' or 'north (A, B)'. Given a set of propositions one can then deduce other relative positions as the induced set of spatial constraints (Dutta 1990). Following an algebraic concept, one does not concentrate on directional relations between points but attempts to find rules for the manipulation of the directional symbols themselves, when combined by operators. An algebra is defined by a set of symbols that are manipulated (here the directional symbols like N, S, E, W), a set of operations and axioms that define the outcome of the operations.

3. Desirable Properties of Cardinal Directions

The intuitive properties of cardinal directions are described in the form of an algebra with two operations applicable to direction symbols:

- the reversing of the direction of travel (*inverse*), and
- the composition of the direction symbol of two consecutive segments of a path (*composition*).

The operational meaning of cardinal direction is captured by a set of formal axioms. These axioms describe the desired properties of cardinal directions and are similar to vector algebra.

3.1. Cardinal Directions

Direction is a binary function that maps two points (P_1, P_2) in the plane onto a symbolic direction d. The specific directional symbols available depend on the system of directions used, e.g., D_4 = {N, E, S, W} or more extensive D_8 = {N, NE, E, SE, S, SW, W, NW}. The introduction of a special symbol *0* (for 'zero') avoids the limitation that direction is only meaningful if the two points are different. In algebra this symbol (usually called *identity*) simplifies the inference rules and increases the deduction power. Its meaning here is that 'two points are too close to determine a direction between them'.

3.2. Reversing direction

Cardinal directions depend on the direction of travel. If a direction is given for a line segment between points P_1 and P_2, the direction from P_2 to P_1 can be deduced (Freeman 1975). This operation is called *inverse* (Figure 3), with

$$inv\ (dir\ (P_1, P_2)\ ) = dir\ (P_2, P_1) \quad \text{and} \quad inv\ (inv\ (dir\ (P_1, P_2)\ )) = dir\ (P_1, P_2)\ .$$

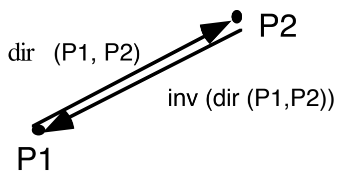

Figure 3: Inverse

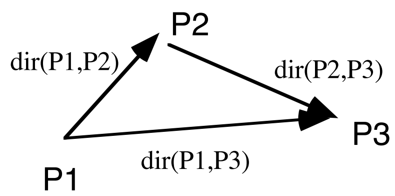

Figure 4: Composition

3.3. Composition

The second operation combines the directions of two contiguous line segments, such that the end point of the first direction is the start point of the second one (Figure 4).

$d_1 \infty d_2 = d_3$ means $dir (P_1,P_2) \infty dir (P_2, P_3) = dir (P_1, P_3)$

Associativity: Compositions of more than two directions should be independent of the order in which they are combined:

$d_1 \infty (d_2 \infty d_3) = (d_1 \infty d_2) \infty d_3 = d_1 \infty d_2 \infty d_3$ *(associative law)*

Identity: Adding the direction from a point to itself, $dir(P_1,P_1) = 0$, to any other direction does not change it.

Algebraic definition of inverse: In algebra, an inverse to a binary operation is defined such that a value, combined with its inverse, results in the identity value. Geometrically the inverse is the line segment that combines with a given other line segment to lead back to the start (Figure 5).

$inv (d) \infty d = 0$ *and* $d \infty inv (d) = 0$

Computing the composition of two directions, where one is the inverse of the other, is in the general case an approximation. The degree of error depends on the definition of *0* and the difference in the distance between the points - if they are the same, the inference rule is exact. This represents a chain of reasoning like: New York is east of San Francisco, San Francisco is west of Baltimore, thus New York is too close to Baltimore (in the frame of reference of the continent) to determine the direction. In general one observes that deduction based on directions alone are only applicable if all the distances are of the same order of magnitude.

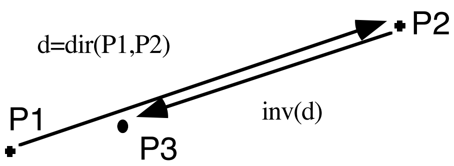

Figure 5: $d \infty inv (d) = 0$

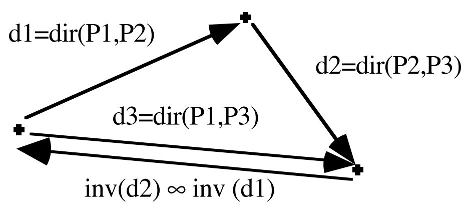

Figure 6: $inv(d_3)= inv(d_1) \infty inv(d_2)$

The inverse is also used to compute the completion of a path to another given one (Figure 6) and one finds that a combined path is piece-wise reversible $inv (d_1 \infty d_2) = inv (d_2) \infty inv (d_1)$.

Idempotent: If one combines two line segments with the same direction, one expects that the result maintains the same direction. This is transitivity for a direction relation:

from d (A,B) and d (B,C) follows d(A,C)

3.4. Definition of Euclidean Exact Reasoning

These rules for qualitative spatial reasoning are compared with the quantitative methods using analytical geometry, as, for example, vector addition. The following definition gives a precise framework for such a comparison. A qualitative rule is called *Euclidean exact,* if the result of applying the rule is the same as if the data was translated to analytical geometry and applied to the equivalent functions, i.e., if there is a homomorphism. Otherwise, it is called *Euclidean approximate* (the symbol + denotes vector addition)

$$dir(P_1,P_2) \infty dir(P_2, P_3) = dir((P_1, P_2) + (P_2, P_3))$$

3.5. Summary of Properties of Cardinal Directions

The basic rules for cardinal directions and the operations of inverse and composition are:

- The composition operation is associative (1').
- The direction between a point and itself is a special symbol *0*, called *identity* (1) (2')
- The combination of an direction and its inverse is *0* (cancellation rule) (3')
- The direction between a point and another is the *inverse* of the direction between the other point and the first (2).
- Combining two equal directions results in the same direction (*idempotent*, transitivity for direction relation) (3).
- The composition can be inverted (4).
- Composition is piece-wise invertible (5).

$$d \infty (d \infty d) = (d \infty d) \infty d \qquad (1')$$

$$d \infty 0 = 0 \infty d = d \qquad (2')$$

$$d \infty inv(d) = 0 \qquad (3')$$

Table 1: Group properties

$$dir(P_1, P_1) = 0 \qquad (1)$$

$$dir(P_1, P_2) = inv(dir(P_2, P_1)) \qquad (2)$$

$$d \infty d = d \qquad (3)$$

for any a, b in D exist unique x in D such that

$$a \infty x = b \text{ and } x \infty a = b \qquad (4)$$

$$inv(a \infty b) = inv(a) \infty inv(b) \qquad (5)$$

Table 2: Desirable properties of directions

Several of the properties of directions are similar to properties of algebraic groups or follow immediately from them. Unfortunately, the idempotent rule (transitivity for direction relation) (3) or the cancellation rule (3') are in contradiction with the remaining postulates. Searching for an inverse x for any $d \infty x = 0$, both $x = d$ (using (3)) or $x = 0$ (using 3') are possible, which contradicts the uniqueness of x in (4). A contradiction results also with associativity (1'), computing d ∞□d ∞ inv (d), which yields either d ∞ inv (d) = *0* or d ∞ *0* = d, depending how parentheses are set.

The customary method to avoid these problems is to define the composition operation to yield a set of values (not a single value as above) and to include in the result set all values that might apply. The cancellation rule (3') then becomes d ∞ inv (d) = {d, 0, inv(d)}, because depending on the distances involved, any of these value may be correct.

The differences are minor and are probably a reflection of the inherent vagueness of the concept of directions (Herskovits 1986, p. 192). Following the concept of prototype values in radial categories (Lakoff 1987), the most likely value is selected and the reasoning rule labeled as Euclidean approximate. The results obtained are exact if the distances involved are all the same (or similar), which is the case for certain situations of navigation of large-scale spaces, e.g., walking in an inner city environment, where the size of the block faces are comparable in length.

4. Two Examples of Systems of Cardinal Directions

Two examples of systems of cardinal directions are studied, both using the same set of eight directional symbols (plus the identity). One is based on cone-shaped (or triangular) areas of acceptance, the other is based on projections. These two semantics for cardinal direction can be related to Jackendoff's principles of centrality, necessity and typicality (Jackendoff 1983, p. 121) as pointed out by Peuquet (1988).

4.1. Directions in 8 or More Cones

Cardinal directions relate the angular direction between a position and a destination to some directions fixed in space. An angular direction is assigned the nearest named direction which results in cone-shaped areas for which a symbolic direction is applicable. This model has the property that "the area of acceptance for any given direction increases with distance" (Peuquet and Zhan 1987, p. 66, with additional references) and is sometimes called 'triangular'.

The set of directional symbols for this system is V_9 = *{N, NE, E, SE, S, SW, W, NW, 0}.*

and a turn of an eighth anti-clockwise is defined as:

e (N) = NE, e (NE) = E, e (E) = SE, ... , e (NW) = N, e (0) = 0.

Eight turns of 45° being the identity function. The inverse is defined as 4 eighth turns inv (d) = e^4 (d). The rules for composition of directions are(2') (3) and the approximation rule (3'). This allows deduction for about one third of possible composition. A set of averaging rules, which allow the composition of directional values which are apart by 1 or 2 eighth turns, is necessary to complete the system. For example N is combined with NE to yield N.

e (e (d)) ∞ d = e (d), e (d) ∞ d = d, e (d) ∞ inv (d) = 0, d ∞ e (d) = d, etc.

In this system, from all the 81 pairs of values (64 for the subset without 0) compositions can be inferred, but most of them only approximately.

	N	NE	E	SE	S	SW	W	NW	0
N	N	n	ne	o	o	o	nw	nw	N
NE	n	ne	ne	e	o	o	o	n	NE
E	ne	ne	E	e	se	o	o	o	E
SE	o	e	e	SE	se	s	o	o	SE
S	o	o	se	se	S	s	sw	w	S
SW	o	o	o	s	s	SW	sw	w	SW
W	nw	o	o	o	sw	sw	W	w	W
NW	n	n	o	o	o	w	w	NW	NW
0	N	NE	E	SE	S	SW	W	NW	0

Table 3: Composition of cone-shaped directions (lower case denotes Euclidean approximate inference).

Directions so defined do not fulfill all the requirements, because they violate the associative property; for example

$(S \infty N) \infty E = 0 \infty E = E$ *but*

$S \infty (N \infty E) = S \infty NE = 0.$

4.2. Cardinal Directions Based on Projections

4.2.1. Directions in 4 half-planes

One can define four directions such that they are pair-wise opposites (Peuquet and Zhan 1987, p. 66) and each pair divides the plane into two half-plains. The direction operation assigns for each pair of points a composition of two directions, e.g., South and East for a total of 4 different directions (Figures 7 and 8).

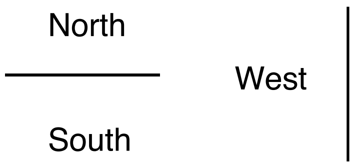

Figure 7: Two sets of half-planes.

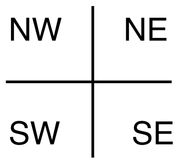

Figure 8: Directions defined by half-planes.

Another justification for this definition of cardinal direction is found in the structure geographic longitude and latitude imposes on the globe. Cone-shaped directions better represent the direction of 'going toward', whereas the 'half-planes' better represent the relative position of points on the earth. Frequently, the two coincide.

In this system, the two projections can be dealt with individually. When the two projections in N-S and E-W are combined to form a single system with the directional symbols:

$V_4 = \{ NE, NW, SE, SW\}$

only the trivial cases (NE ∞ NE = NE, etc.) can be resolved.

4.2.2. Projection-Based Directions with Neutral Zone

If points which are near to due north (or west, east, south) are not assigned a second direction, i.e., one does not decide wether or not such a point is more east or west, one divides effectively the plane effectively in 9 regions

(Figure 9), a central neutral area, four regions where only one direction applies and 4 regions where two are used. For N-S the three values for direction d_{ns} are {N, P, S} and for the E - W direction the values are d_{ew} {E, Q, W}.

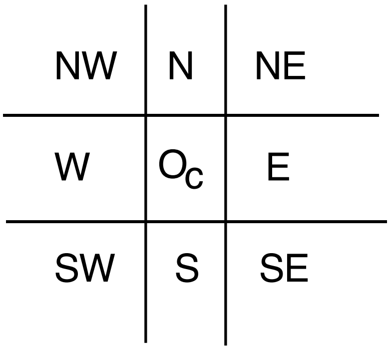

Figure 9: Directions with neutral zone

It is important to note that the width of the 'neutral zone' is not defined a priori. Its size is effectively decided when the directional values are assigned and a decision is made that P_2 is north (not north-west or north-east) of P_1. The algebra deals only with the directional symbols, not how they are assigned. The system assumes that these decisions are consistently made in order to determine if a deduction rule is Euclidean exact or not.

In one projection, selecting d_{ns}= {N, P, S} as example, the inverse operation is defined as inv (N) = S, inv (S)= N, inv (P) = P. For the composition the rules (3), (2') and (3') are required. The two projections in N-S and E-W are then combined to form a single system, in which for each line segment one of 9 compositions of directions are assigned.

V_9 = {NE, NQ, NW, PE, PW, PQ, SE, SQ, SW}

this can be simplified to yield the customary cardinal directions, if P and Q are eliminated (replacing PQ by 0)

V_9 = {NE, N, NW, E, W, 0, SE, S, SW}

If any of the result for the two projections is approximate the total result is considered approximate reasoning.

The inverse is defined as the combination of the inverses of the projection, and the composition is the combination of the compositions of each projection. Using the three rules the values for each composition are computed.

	N	NE	E	SE	S	SW	W	NW	0
N	N	N	NE	e	o	w	NW	NW	N
NE	NE	NE	NE	e	e	o	n	n	NE
E	NE	NE	E	SE	SE	s	o	n	E
SE	e	e	SE	SE	SE	s	s	o	SE
S	o	e	SE	SE	S	SW	SW	w	S
SW	w	o	s	s	SW	SW	SW	w	SW
W	NW	n	o	s	SW	SW	W	NW	W
NW	NW	n	n	o	w	w	NW	NW	NW
0	N	NE	E	SE	S	SW	W	NW	0

Table 4: Composition of projection-based directions (lower case denotes Euclidean approximate inference)

4.3. Assessment

The power of the 8 direction cone-shaped and the 4 half-plane based directional system are similar. Each system uses 9 directional symbols, 8 cone-shaped directions plus identity on one hand, the Cartesian product of 3 values (2 directional symbols and 1 identity symbol) for each projection on the other hand. The same rules are used to build both systems, but the cone-shaped system had to be completed with a number of 'averaging rules'.

Introducing the identity symbol *0* increases the number of deduction in both cases considerably. Only 8 out of 64 compositions could be resolved for cone-shaped directions and only trivial cases can be resolved for the half-plane based system.The systems with identity allow conclusions for any pair of input values (81 different pairs) at least approximately. The comparison reveals, that the projection-based directions result in less cases of approximation results than the cone-shaped (32 vs. 56) and that it yields less often the value *0* (9 vs. 25). Considering the definite values (other than 0) deduced, differences appear only for results where the 'averaging rules' are used for the cone-shaped directions, and the values differ by one eighth turn only. In conclusion, the projection-based system is producing more exact and more definite results.

An implementation of these rules and comparing the computed combinations with the exact value was done and confirms the theoretical results. Comparing all possible 10^6 compositions in a grid of 10 by 10 points (with a neutral zone of 3 for the projection-based directions) shows that the results for the projection-based directions are correct in 50% of cases and for cone-shaped directions in only 25%. The result *0* is the outcome of 18% of all cases for the projection-based, but 61% for the cone-shaped directions. The direction-based system with an extended neutral zone produces in only 2% of all cases a result that is a quarter turn off, otherwise the deviation from the correct result is never more than one eighth of a turn, namely in 13% of all cases for cone-shaped and 26% for projection-based direction systems. In summary, the projection-based system of directions produces in 80% of all cases a result that is within 45° and otherwise the value *0*.

5. Spatial Reasoning Studies

Cardinal directions are similar to the four directions 'Front', 'Back', 'Left', and 'Right', which one encounters in spatial references (Herskovits 1986, Retz-Schmidt 1987). Although cardinal directions are prominent in geography, humans often use local reference frames, based on either their own position and heading or an implied frame for the object of interest. It is then necessary to convert distance and direction expressions valid in one frame of reference to the expression valid in another frame (McDermott gives a solution based on coordinate values including some 'fuzz' (1984)). In a qualitative reasoning system, composition is used.

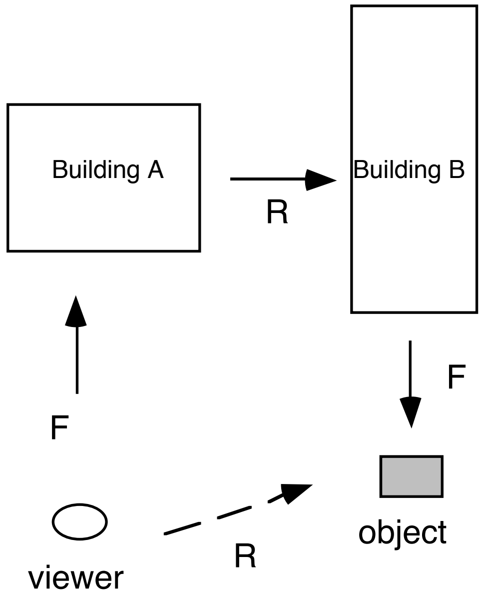

Figure 10: Reference frame example

The object in figure 10 is in front of Building B which is to the right of Building A that the viewer is facing. Using composition of direction relations (Table 4) we can conclude that the object is to the right of the viewer using the following chain of inferences:

c1 = F (front)

c2 = R (right)

*c'3 = F, with ref3 = e**4 (the reference frame of Builldng B is 4 eighths turned)*

c1 ∞ c2 = F ∞ R = FR

*c3 = e**4 (F) = B*

c1 ∞ c2 ∞ c3 = FR ∞ B = R (using for example NE ∞ S = E in Table 4).

Conclusion: the object is to the right of the viewer, which is correct.

Artificial intelligence has found the formalization of common sense knowledge a formidable task (Lenat, et al. 1990). McCarthy (1975) proposed a subdivision of endeavors and singled out spatial reasoning as an important task, prevalent in many other kinds of reasoning. Spatial reasoning has been further subdivided in different approaches, mostly concentrating on the aspects necessary to resolve some specific tasks.

Most relevant for geographic or large-scale space is Kuipers' work, which primarily dealt with aspects of knowledge acquisition for wayfinding in large-scale space and to a lesser degree with spatial reasoning in a restricted sense (Kuipers 1982, Kuipers 1983, Kuipers 1990a, Kuipers 1990b, Kuipers and Levit 1990). McDermott is more concerned with navigation in a complex environment, consisting of buildings etc. and deductions of routes that permit physical passage(1980 1984). Davis deals with the acquisition of geographic (i.e. large-scale) spatial information for navigation purposes (Davis 1986).

Others have primarily studied vision, and concentrated on aspects of understanding images for which spatial reasoning is necessary. An entirely qualitative approach is the work on *symbolic projections*. It translates exact metric information, primarily about objects in pictures, into a qualitative form (Chang, et al. 1990). Segments in pictures are projected vertically and horizontally, and their order of appearance is encoded in two strings. Spatial reasoning, especially spatial queries, are executed as fast substring searches (Chang, et al. 1987). Holmes and Jungert (1992) have demonstrated how symbolic projections can be applied to knowledge-based route planning in digitised maps. Papadias and Sellis (1993) have combined the concepts of symbolic projections and topological reasoning in one representation framework.

Related to composition of spatial relations is the problem of *consistency checking* in spatial constraint networks. A spatial constraint network is a qualitative graph-based description of a scene, where the nodes represent objects and the arcs sets of spatial relations corresponding to disjunctions of possible relations between objects. Inserting a new relation between two objects in the network affects not only the two objects, but because of composition, the insertion might yield additional constraints between other objects in the network (*constraint propagation*). Studies of constraint propagation and consistency checking in networks of spatial relations can be found in (Guesgen 1989) and (Hernández 1993).

In Cognitive Science and Psycology the study of spatial relations and their use in human reasoning has been a topic of interest for many years. Stevens and Coupe (1978) observed distortions in the recall of direction relations between cities in Northern America and they attributed these findings to a hierarchical representation of space. Hirtle and Jonides (1985), observed similar distortions when people reason about distance relations. Tversky (1993) proposed *cognitive maps* (i.e., map-like mental constructs which can be mentally inspected), *cognitive collages* (i.e., thematic overlays of multimedia from different points of view) and *spatial mental models* (i.e., representations which capture spatial relations among distinct objects without preserving metric information) as alternative mental representations that depend on the reasoning tasks to be performed. Huttenlocker (1968) suggested that people construct spatial representations in order to draw conclusions about spatial or non-sparial tasks.

Several computational models and representation schemes have been proposed to deal with aspects of spatial reasoning. Glasgow and Papadias (1992), for instance, incorporate hierarchical spatial representations that can be used to reason with spatial relations in their representation scheme of *computational imagery*. Randell et al. (1992) proposed a theory for topological reasoning expressed in a many-sorted logic. Levine (1978) developed a semantic network where the arcs encode spatial relations such as left, inside etc. Myers and Konolige, (1992) designed a hybrid representation system that integrates both deductive and non-deductive spatial reasoning. A study of several systems that have been used to represent and reason with spatial relations can be found in (Papadias and Kavouras 1994). Although the different systems can represent the same information about a given domain (they can be made *informationally equivalent*), they are not *computationally equivalent* since the efficiency of the reasoning mechanisms is not the same. Identical tasks may involve different algorithmic solutions and consequently have different complexities in distinct representational systems. A discussion about *informational* and *computational equivalence* of spatial representations can be found in (Larkin and Simon 1987).

6. Research Envisioned

This section describes a comprehensive set of investigations to formalize spatial reasoning with large-scale space. The description links different aspects of spatial reasoning and discusses their relations. The research effort concentrates on the formalization of spatial reasoning using methods similar to those used above to reason about cardinal directions. Its goals are more modest than the ones of the researchers interested in navigation or vision, intending only to formalize inference rules relevant for spatial relations in large-scale space. This is similar to efforts of Dutta (1990). Despite these restrictions, a sufficiently complex problem remains to be investigated with results of utility for GIS.

Following the classical definition by Felix Klein, geometry is the branch of mathematics that investigates properties of configurations which remain invariant under a group of transformations (Blumenthal and Menger 1970). One differentiates the topological relations like 'inside', 'connected', 'bounding' etc., which remain invariant under homomorphic transformations, from metric relations, like 'distance', 'direction' etc., which remain invariant only under the much more restricted group of rotations and translations.

Initial work investigating topological structures (Egenhofer 1989, Egenhofer and Herring 1990, Egenhofer and Franzosa 1991) and metric relations, specifically distance and cardinal directions (Frank 1990a, Frank 1991) indicates that topological relations are the first level qualification which are further described by metric relations. This reflects the fact that topological relations are invariant under a much larger group of transformations than metric relations. Topological relations can be observed directly and characterized with a value on a nominal scale.

A method to characterize topological relations was proposed by Egenhofer (1990). It uses ideas from Allen (1983), initially developed for reasoning about time intervals, and applies them to two dimensional space. It introduces a simple characterization of topological relations in terms of 3 primitives, namely the test if the intersection of boundary, interior and complement of the two figures is empty.

The plan is stressing a stepwise extension from the known, qualitative reasoning with cardinal directions and topological relations. Each step is directed towards maximum separation of concerns - assuming that only the most simple problems can be solved (a tendency that can be generally observed in AI research). The results are then combined, to yield the complex systems used by human beings - the complexity being the result of the combination, not being present in the single step.

6.1. Distance and Directions between Points Embedded in 2 Dimensional Space

Distance and cardinal directions are two very widely used metric relations in large-scale space. Indeed, cardinal directions are only used in large-scale space. The prototypical case for the metric relations are distance and directions between points in the plane, i.e. co-dimension 2 (0 D points embedded in 2 D space).

6.1.1. Formalization of qualitative reasoning with distances.

Preservation of metric properties is typical for our experience with physical, solid objects in small scale space (Adler, et al. 1965). They can be moved but preserve length, angles etc. and Euclidean geometry can be seen as the ideal formalization of these concepts. In real physical space not all the assumed properties of Euclidean geometry are fulfilled. For example, one cannot insert between any two (physically embodied) points another one. Furthermore, movements back and forth over the same distance do not exactly cancel.

Similarly, visual perception cannot distinguish between two points that are sufficiently close (Roberts and Suppes 1967, Zeeman 1962). *Measurement theory* provides a theoretical base when the problems of limited acuity (the just noticeable difference) are taken into account (Krantz, et al. 1971, Scott and Suppes 1958). Based on problems with observations of economic utility, Luce (1956, p. 181) has defined semiorders as

> "Let S be a set and < and ~ be two binary relations defined over S . (< , ~) is a semiordering of S if for every a, b, c, and d in S the following axioms hold:
>
> S1. exactly one of a < b, b < a, or a ~ b obtains,
>
> S2. a ~ a.
>
> S3. a < b, b ~ c, c < d imply a < d,
>
> S4. a < b, b < c, b ~ d imply not both a ~ d and c ~ d."

These observations can be visualized using graphs (Roberts 1969, Roberts 1971) and lead to tolerance geometry (Robert 1973). It will be necessary to consider these results when defining qualitative metric relations, using expressions like 'near', 'far' or 'very far' (Frank 1991).

6.1.2. Integrating reasoning about distances and directions into a single set of inference rules.

Combining distance and direction reasoning, one becomes aware of the limitations of separating distance and direction reasoning in individual chains of inference. The integration of distance and direction inferences increases the precision of the results. Separating distance and direction information and applying independent rules for the inference on distance and direction, ignores the potential influence between distance and direction inference and reveals the shortcomings of both distance and direction reasoning.The result of far north composed with near south is far north (and not *0* as would result from rule (3') above).

It is possible to construct tables for the addition of qualitative distance symbols under the assumption that the two paths are anti-parallel (i.e., parallel, but in the inverse direction). Other tables can be constructed if the two paths form a right angle, or 1 or 3 quarter turns apart. Such distance reasoning rules take into account the angle between the two paths, computed as the (symbolic) difference between the qualitative directions. The same concept is applied to the direction reasoning. Hong (1994) produced tables similar to Table 4 indexed by the difference in length of the paths (computed from the qualitative distance information available).

This method improves the precision, without having to deal with the composition of all possible inputs (as suggested in (Mukerjee 1990)). It takes advantage of the invariance of composition of paths under rotation. In lieu of a table for the results of all input values, only a small set of smaller tables are necessary.

6.2. Combining reasoning about distance and direction relations with topological relations

A valid criticism to formalized spatial reasoning, primarily if using a coordinate plane, is, that it attempts to map all information given into a single, uniform spatial frame (for examples see (McDermott and Davis 1984)). This does not compare with the reasoning methods of human beings, which resolves spatial relations in groups (Stevens and Coupe 1978). It is intuitively clear, that spatial reasoning for a trip from home to office can be separated in several steps, starting with navigating inside one's home, moving with the car through the street network, moving inside the office building etc.

In a first instance, the 'inside' topological relation is used, because it allows for propagation of spatial relations from the container to the parts contained. For example, if the distance between Maine and California is known to be about 4000 miles, one may conclude that the distance between any point in Maine and a point in California is in the same order of magnitude. Further inside is transitive (A inside B and B inside C includes A inside C).

6.2.1. Hierarchically regionalized space

A multi-level hierarchical structure of areas each containing smaller areas is an extreme simplification but leads to a straight-forward formalization. A similar simplification has been used successfully by Christaller for his central place theory (1966). In a hierarchically regionalized space, reasoning about distances and cardinal directions between points is enormously simplified. Papadias and Sellis (1992) have shown how inference of direction relations can be achieved using hierarchical representations of space, while Car and Frank (1994) developed algorithms for hierarchical wayfinding. The errors produced in hierarchical systems resemble the ones observed in humans performing similar operations.

6.2.2. General partially ordered regionalized space

In general, the inclusion relation orders regions (i.e. two dimensional areas) partially, but does not automatically produce a strict hierarchy or even a lattice (Kainz 1988, Saalfeld 1985). The rules for reasoning about distances and directions developed for the hierarchical case must be extended to cope with situations, where a point is contained in multiple regions (town, school district, watershed, etc.), which do partially overlap. Papadias et al., (1994) have dealt with aggregation hierarchies where a geographic entity may belong to more than one parent and observed that the improper specification of multiple hierarchies can lead to the inheritance of incosistent relations.

6.3. Generalize distance and direction relations to extended objects

Distance and direction relations between extended objects, i.e. objects, which are not point-like, need to be defined as an extension of metric relations between points. Humans ask questions like, what is the direction from Maine to Canada and expect an answer of 'north' (indeed, from Maine, Canada is also east and west, and there are some points in Maine where Canada is also found to the south). Peuquet and Zhan (1987) developed algorithms for the determination of direction relations based on a 'visual interpretation' but their determinations do not always correspond with others' judgment. Papadias and Sellis (1994) have also dealt with direction relations between arbitrary objects represented by their minimum bounding rectangles. Their method is projection-based, unlike Peuquet's method which is based on the concept of cone-directions.

Distances and directions are easily definable only for extended objects which are disjoint - i.e. the topological relations provide the first level of classification of spatial relations, and the metric relations a second level of characterization, with distance and directions only applicable for disjoint objects (different from the approach by Peuquet and Zhan).

Using the results from the formalization of distance and direction between point like objects and the solutions to extending topological relations to higher co-dimensions, distance and direction relations between extended objects must be formalized, explaining distances between lines and points, two lines, lines and areas ('how far is the town of Dixmont from the Interstate'), etc.

6.4. Finding other metric relations to characterize relations of objects

Not only the situation of topologically disjoint objects can be further characterized with a metric value. Cardinal direction is also used for objects which are inside others. For example, Aroostook County (a county in the state of Maine) is said to be in the northern part of Maine - characterizing the topological relation of 'inside' additionally with a direction term. This is a different use of a cardinal direction than between two points.

Generalizing this example, the following device helps to discover additional metric relations. Imagine two objects, first quite distant from each other (figure 11a), characterized by disjoint and a distance value. The smaller object approaches the other and 'meet' (figure 11b), characterized by the topological relation alone, which implies distance 0. Moving the smaller object further, the objects overlap (figure 11c), which could be characterized by a metric value, indicating the degree of overlap. Then the objects meet from the inside for a moment (figure 11d), before the one is inside the other(figure 10 e).

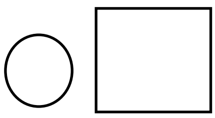

Figure 10 a: Disjoint

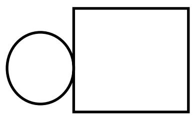

Figure 10 b: Meet

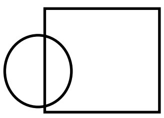

Figure 10 c: Overlap

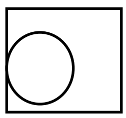

Figure 10 b: Covered

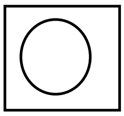

Figure 10 e: Inside

Egenhofer and Al-Taha (1992) studied the gradual changes of topological relations for several cases of transfomations involving translations, rotations, expansions and reductions. Some topological relations are 'sharp' in the sense that they apply only for a single moment when an object moves, whereas others are 'wide' and apply for some range of situations, which can be further characterized with a metric value. In English one can say

Kansas is further inside the United States than West Virginia.

Object A is 'barely' overlapping B in figure y.

Both expressions indicating that a metric characterization of the topological relation is appropriate.

The topological relations which are further characterized by a metric relation are, in first line, 'overlap' and 'inside' (but also 'meet along boundary' - characterized by the length of the common boundary). Possible candidates for a metric relation would be the relation of area inside and outside for overlap and a similar measure to determine if A is more inside than B.

7. Conclusion

Qualitative spatial reasoning is prevalent in human thinking about space and spatial situations. Its formalization almost always uses Euclidean geometry and the Cartesian coordinate system. This is admittedly not adequate to model human cognitive structure, but other solutions are not currently known. Spatial reasoning is crucial for progress in Geographic Information Systems, primarily, but not only, for the design and the programming of spatial query languages. Spatial relations depend on the type of space, for example cardinal directions are only used in large-scale spaces. The focus in this paper is on large-scale or geographic space, but the problems identified, the approaches, the methods and some of the results are meaningful for other types of spaces.

This paper introduced a system for qualitative spatial reasoning with cardinal distances from an algebraic point of view. Two operations, *inverse* and *composition* are applied to direction symbols and their meanings are formalized with a set of axioms. Three requirements, that directions should fulfill, are

- the direction from a point to itself is a special value, meaning 'too close to determine a direction',
- every direction has an inverse, namely the direction from the end point to the start point of the line segment, and
- the composition of two line segments with the same direction results in a line segment with the same direction.

In order to evaluate the expressive power of our qualitative reasoning approach, the notion of 'Euclidean exact' was defined using a homomorphism. A deduction rule is called 'Euclidean exact' if it produces the same results as Euclidean geometry operations would. Two different specific systems for cardinal directions, one based on cone-shaped regions the other based on projections, are explored. They produce very similar outputs, but the projection-based cardinal directions yield more often results which are 'Euclidean exact'.

Cardinal directions are used as an example to show the advantages of the algebraic approach over the traditional propositional one. A series of research steps is laid out, which will lead to a comprehensive set of formalized spatial relations, that interact in a controlled way. Several guiding principles for the organization of such work are possible. The taxonomy of spaces from (Couclelis and Gale 1986) and the properties of the dominant operations could be used. A linguistic approach would primarily investigate which spatial relations are used in which context (e.g. cardinal directions are typical for large-scale spaces).

The guideline selected is based on traditional divisions in mathematics. The characterization of a spatial situation is done first by assessing the topological relations and then the metric properties and the same idea is followed to organize the investigations envisioned. A series of steps are listed, each addressing a specific researchable problem:

- Extension of topological relations to points, lines and areas (co-dimension > 0)
- Qualitative reasoning with distances, and integration of reasoning about distances and directions,
- Reasoning in regionalized spaces, supporting distance and direction reasoning using the topological relation 'inside',
- Generalize distance and direction relations to extended (non-point) objects, and
- Find additional metric relations which characterize topological relations other than 'disjoint'.

The guiding expectation is, that the combination of different methods of qualitative spatial reasoning will provide the accuracy expected, an accuracy that cannot be found in any single method in isolation. The lead example is reasoning about distances and directions in isolation, which leads to approximate rules like 'cancelation' (3'), which are obviously often inappropriate. However, the combination of qualitative distance and direction reasoning improves precision of inferences enormously.

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References

D. Adler, M. Bazin and M. Schiffer. 1965. *Introduction to General Relativity*. New York: McGraw Hill.

J. F. Allen. 1983. Maintaining Knowledge about Temporal Intervals. *Communications of the ACM* 26 : 832-843.

L. M. Blumenthal and K. Menger. 1970. *Studies in Geometry*. San Francisco: Freeman and Company.

D. G. Bobrow, S. Mittal and M. J. Stefik. 1986. Expert Systems: Perils and Promise. *Communications of the ACM* 29 (9) : 880-894.

Campari I., A.U. Frank (1993), Cultural differences in GIS: A basic approach. Proceeding of EGIS'93, Genua, March 28- April 1, pp.1-10

Campari I., A.U. Frank (1994), Cultural aspects and cultural differences in GIS. In T. Nyerges (Ed.), Human Computer Interaction for GIS, in press

I. Campari. 1991. "Some Notes on GIS: The Relationship between their Practical Application and their Theoretical Evolution". In *Spatial Languages*. Edited by D. M. Mark and A. U. Frank. in preparation. Amsterdam: Kluver.

A. Car and A. Frank. 1994. General Principles of Hierarchical Spatial Reasoning: The Case of Wayfinding. To appear in the *Proceedings of the 6th International Symposium on Spatial Data Handling*, Endinburgh, U.K. Taylor Francis 1994.

S.-K. Chang, E. Jungert and Y. Li. 1990. "The Design of Pictorial Databases Based Upon the Theory of Symbolic Projection". In *Design and Implementation of Large Spatial Databases*. Edited by A. Buchmann, O. Günther, T. R. Smith and Y.-F. Wang. 303 - 324. New York NY: Springer Verlag.

S. K. Chang, Q. Y. Shi and C. W. Yan. 1987. Iconic Indexing by 2-D String. *IEEE Transactions on Pattern Analysis and Machine Intelligence* 9 (3) : 413 - 428.

W. Christaller. 1966. *Central Places in Southern Germany*. Englewood Cliffs, NJ: Prentice Hall.

H. Couclelis and N. Gale. 1986. Space and Spaces. *Geografiska Analer* 68 B : 1 - 12.

H. Couclelis. 1992. People Manipulate Objects (but Cultivate Fields): Beyond the Raster-Vector Debate in GIS. In Frank, A.U., Campari, I., Formentini, U. (eds.) *Proceedings of the International Conference GIS - From Space to Territory: Theories and Methods of Spatio-Temporal Reasoning in Geographic Space*. Pisa, Italy, September 1992. Springer-Verlag.

E. Davis. 1986. *Representing and Acquiring Geographic Knowledge*. Research Notes in Artificial Intelligence. Los Altos, CA: Morgan Kaufmann.

J. de Kleer and J. S. Brown. 1985. "A Qualitative Physics Based on Confluence". In *Formal Theories of the Commonsense World*. Edited by J. R. Hobbs and R. C. Moore. 109 - 184. Norwood NJ: Ablex Publishing Corp.

S. Dutta. 1988. "Approximate Spatial Reasoning". In *First International Conference on Industrial and Engineering Applications of Artificial Intelligence and Expert Systems*. 126 - 140. Tullahoma, Tennessee: ACM Press.

S. Dutta. 1990. "Qualitative Spatial Reasoning: A Semi-quantitative Approach Using Fuzzy Logic". In *Design and Implementation of Large Spatial Databases*. Edited by A. Buchmann, O. Günther, T. R. Smith and Y.-F. Wang. 345 - 364. New York NY: Springer Verlag.

M. Egenhofer and J. R. Herring. 1990. "A Mathematical Framework for the Definition of Topological Relationships". In *4th International Symposium on Spatial Data Handling*. Edited by K. Brassel. 803-813. Zurich, Switzerland: International Geographical Union IGU, Commission on Geographic Information Systems.

M. J. Egenhofer. 1989. "A Formal Definition of Binary Topological Relationships". In *Third International Conference on Foundations of Data Organization and Algorithms (FODO)*. Edited by W. Litwin and H.-J. Schek. 457-472. Paris, France: Springer-Verlag, Berlin, Heidelberg Germany(FRG).

M. J. Egenhofer. 1991. Interacting with Geographic Information Systems via Spatial Queries. *Journal of Visual Languages and Computing* 1 (4) : 389-413.

M. J. Egenhofer and R. Franzosa. 1991. Point-Set Topological Spatial Relations. *International Journal of Geographical Information Systems*, 5.

M. J. Egenhofer and K Al-Taha. 1992. Reasoning about Gradual Changes of Topological Reasoning. In Frank, A.U., Campari, I., Formentini, U. (eds.) Proceedings of the International Conference GIS - From Space to Territory: Theories and Methods of Spatio-Temporal Reasoning in Geographic Space. Pisa, Italy, September 1992. Springer-Verlag.

A. U. Frank. 1990a. "Qualitative Spatial Reasoning about Cardinal Directions". In *Autocarto 10*. Edited by D. M. Mark and D. White. Baltimore: ACSM/ASPRS.

A. U. Frank. 1990b. "Spatial Concepts, Geometric Data Models and Data Structures". In *GIS Design Models and Functionality*. Edited by D. Maguire. Leicester, UK: Midlands Regional Research Laboratory, University of Leicester.

A. U. Frank. 1991. Qualitative Spatial Reasoning about Distances and Directions in Geographic Space. *Journal of Visual Languages and Computing*, 3, pp 343-371.

A. U. Frank and D. M. Mark. 1991. "Language Issues for Geographical Information Systems". In *Geographic Information Systems: Principles and Applications*. Edited by D. Maguire, D. Rhind and M. Goodchild. London: Longman Co. (in press).

J. Freeman. 1975. The modelling of spatial relations. *Computer Graphics and Image Processing* 4 : 156-171.

Freska. 1991. "Qualitative Spatial Reasoning". In *Spatial Languages*. Edited by D. M. Mark and A. U. Frank. in preparation. Amsterdam: Kluver.

J.I. Glasgow and D. Papadias. 1992. Computational Imagery. *Cognitive Science*, 16, 355-394.

H.W. Güsgen 1989. Spatial Reasoning Based on Allen's Temporal Logic. TR-89-049, International Computer Science Institute, Berkeley 1989.

D. Hernàndez 1993. Maintaining Qualitative Spatial Knowledge. In Frank, A.U., Campari, I. (eds.) Proceedings of the *European Conference on Spatial Information Theory COSIT*, Elba, Italy, September 1993. Springer-Verlag.

A. Herskovits. 1986. *Language and Spatial Cognition - An Interdisciplinary Study of the Propositions in English*. Stiudies in Natural Language Processing. Cambridge UK: Cambridge University Press.

S. Hirtle and J. Jonides. 1985. Evidence of Hierarchies in Cognitive Maps. *Memory and Cognition* 13, 208-217.

P.D Holmes and E. Jungert. 1992. Symbolic and Geometric Connectivity Graph Methods for Route Planning In Digitized Maps. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. PAMI-14, no 5, pp. 549-565.

J-H. Hong. 1994. Qualitative Distance and Direction Reasoning in Geographic Space. Unpublished Ph.D. Thesis, Department of Surveying Engineering, University of Maine, Orono.

J. Huttenlocker. 1968. Constracting Spatial Images: A Strategy in Reasoning. *Psychological Review*, 4, 277-299.

R. Jackendoff. 1983. *Semantics and Cognition*. Cambridge, Mass.: MIT Press.

W. Kainz. 1988. "Application of Lattice theory to geography". In *Third International Symposium on Spatial Data Handling*. Edited by D. F. Marble. 135-142. Sydney, Australia: International Geographical Union, Commission on Geographical Data Sensing and Processing.

D. H. Krantz, R. D. Luce, P. Suppes and A. Tversky. 1971. *Foundations of Measurement*. New York: Academic Press.

B. Kuipers. 1982. The 'Map in the Head' Metaphor. *Environment and Behaviour* 14 (2) : 202 -220.

B. Kuipers. 1983. "The Cognitive Map: Could it have been any other way?". In *Spatial Orientation*. Edited by H. L. Pick and L. P. Acredolo. 345 - 359. New York NY: Plenum Press.

B. Kuipers. 1990a. "Commonsense Knowledge of Space: Learning from Experience". In *Advances in Spatial Reasoning*. Edited by S.-s. Chen. 199 - 206. Norwood, NJ: Ablex Publishing Corp.

B. Kuipers. 1990b. "Modeling Spatial Knowledge". In *Advances in Spatial Reasoning*. Edited by S.-s. Chen. 171 - 198. Norwood, NJ: Ablex Publishing Corp.

B. Kuipers and T. S. Levit. 1990. "Navigation and Mapping in Large-Scale Space". In *Advances in Spatial Reasoning*. Edited by S.-s. Chen. 207 - 251. Norwood, NJ: Ablex Publishing Corp.

G. Lakoff. 1987. *Women, Fire, and Dangerous Things: What Categories Reveal About the Mind*. Chicago, IL: University of Chicago Press.

J. Larkin and H.A. Simon. 1987. Why a Diagram is (Sometimes) Worth Ten Thousand Words. Cognitive Science, 11, pp65-99.

D. G. Lenat, R. V. Guha, K. Pittman, D. Pratt and M. Shepherd. 1990. Cyc: Toward programs with common sense. *Communications of the ACM* 33 (8) : 30 - 49.

M. Levine, M. 1978. A Knowledge-Based Computer Vision System. In Hanson, A. and Riseman, E. (eds.) *Computer Vision Systems*. New York: Academic. pp 335-352.

R. D. Luce. 1956. Semiorders and a theory of utility discrimination. *Econometria* 24 : 178 - 191.

D. M. Mark and A. U. Frank. 1990. *Experiential and Formal Representations of Geographic Space and Spatial Relations*. Technical Report (in press). : NCGIA.

D. M. Mark, A. U. Frank, M. J. Egenhofer, S. M. Freundschuh, M. McGranaghan and R. M. White. 1989a. *Languages of Spatial Relations: Initiative Two Specialist Meeting Report*. Technical Report 89-2. : National Center for Geographic Information and Analysis.

D. M. Mark, M. D. Gould and J. Nunes. 1989b. "Spatial Language and Geographic Information Systems: Cross Linguistic Issues". In *II Conferencia Latinoamericana sobre la Tecnologia de los Sistemas de Informacion Geografica*. Edited by R. Ponte, A. Guevara and M. Lyew. 105 - 130. Merida (Venezuela): Universidad de los Andes.

J. McCarthy. 1977. "Epistomological problems of artificial intelligence". In *IJCAI-77*. 1038 - 1044. Cambridge, MA.

D. McDermott. 1980. "A theory of metric spatial inference". In *First National Conference on Artificial Intelligence (AAAI)*. 246-248. :

D. McDermott and E. Davis. 1984. Planning routes through uncertain territory. *Artificial Intelligence* 22 : 107-156.

M. McGranaghan. 1991. "Schema and Object Matching as a Basis for Interpreting Textual Specifications of Geographical Location". In *Spatial Languages*. Edited by D. M. Mark and A. U. Frank. in preparation. Amsterdam: Kluver.

D. Montello. 1993. Scale and Multiple Psychologies of Space. In Frank, A.U., Campari, I. (eds.) Proceedings of the *European Conference on Spatial Information Theory COSIT*, Elba, Italy, September 1993. Springer-Verlag.

K. L., Myers and K. Konolige. 1992. Reasoning with Analogical Representations. In the Proceedings of the 3rd International Conference on Principles of Knowledge Representation and Reasoning, KR 92. Morgan-Kaufman.

A. Mukerjee. 1990. *A Qualitative Model for Space*. TAMU-90-005. Huston, TX: Texas A & M University.

NCGIA. 1989. The U.S. National Center for Geographic Information and Analysis: An overview of the agenda for research and education. *International Journal of Geographical Information Systems* 2 (3) : 117-136.

S. Nirenburg and V. Raskin. 1987. "Dealing with Space in Natural Language Processing". In *Spatial Reasoning and Multi-Sensor Fusion*. Edited by A. Kak and S.-s. Chen. 361 - 370. Pleasan Run Resort, St. Charles, IL: Morgan Kaufmann Publishers.

D. Papadias, A. Frank and M. Koubarakis. 1994. Constraint-Based Reasoning in Geographic Databases: The Case of Symbolic Arrays. In the *Proceedings of the second ICLP Workshop on Deductive Databases*, Italy.

D. Papadias and M. Kavouras. 1994. Acquiring, Representing and Processing Spatial Relations. In the *Proceedings of the 6th International Symposium on Spatial Data Handling*, Endinburgh, U.K. Taylor Francis 1994.

D. Papadias and T. Sellis. 1992. Spatial Reasoning using Symbolic Arrays. In Frank, A.U., Campari, I., Formentini, U. (eds.) *Proceedings of the International Conference GIS - From Space to Territory: Theories and Methods of Spatio-Temporal Reasoning in Geographic Space*. Pisa, Italy, September 1992. Springer-Verlag.

D. Papadias and T. Sellis. 1993. The Semantics of Relations in 2D Space Using Representative Points: Spatial Indexes. In Frank, A.U., Campari, I. (eds.) *Proceedings of the European Conference on Spatial Information Theory COSIT*, Elba, Italy, September 1993. Springer-Verlag.

D. Papadias and T. Sellis. 1994. On the Qualitative Representation of Spatial Knowledge in Two-Dimensional Space. To appear in the *Very Large Data Bases Journal., VLDB Special Issue on Spatial Databases*.

D. Peuquet and C.-X. Zhan. 1987. An algorithm to determine the directional relationship between arbitrarily-shaped polygons in a plane. *Pattern Recognition* 20 : 65-74.

D. J. Peuquet. 1988. "Toward the definition and use of complex spatial relationships". In *Third International Symposium on Spatial Data Handling*. Edited by D. F. Marble. 211-224. Sydney, Australia: International Geographical Union, Commission on Geographical Data Sensing and Processing.

D. Randell, Z. Cui and A. Cohn. 1992. A Spatial Logic Based on Regions and Connection. In the *Proceedings of the 3rd International Conferrence on Principles of Knowledge Representation and Reasoning*, KR 92. Morgan-Kaufman Publishers.

G. Retz-Schmidt. 1987. "Deitic and Intrinsic Use of Spatial Propositions: A Multidisciplinary Comparison". In *Spatial Reasoning and Multi-Sensor Fusion*. Edited by A. Kak and S.-s. Chen. 371 -380. Pleasan Run Resort, St. Charles, IL: Morgan Kaufmann Publishers.

F. S. Robert. 1973. Tolerance Geometry. *Notre Dame Journal of Formal Logic* 14 (1) : 68-76.

F. S. Roberts. 1969. "Indifference Graphs". In *Proof Techniques in Graph Theory*. Edited by F. Harary. 139 - 146. New York: Academic Press.

F. S. Roberts. 1971. On the Compatibility between a Graph and a Simple Order. *Journal of Combinatorial Theory* 11 : 28 - 38.

F. S. Roberts and P. Suppes. 1967. Some Problems in the Geometry of Visual Perception. *Synthese* 17 : 173 - 201.

A. Saalfeld. 1985. Lattice structures in geography: The quadtree and related hierarchical data structures, Computing Surveys. 16 : pp. 482-489 Samet, H. 1984, p. 187.

D. Scott and P. Suppes. 1958. Foundational Aspects of Theories of Measurements. *Journal of Symbolic Logic* 23 (2) : 113 - 128.

A. Stevens and P. Coupe. 1978. Distortions in judged spatial relations. *Cognitive Psychology* 10 : 422 - 437.

J. M. Synge. 1941. *Plays, Poems and Prose*. London: Dent.

W. R. Tobler and S. Wineberg. 1971. A Cappadocian Speculation. *Nature* 231 (May 7) : 39-42.

B. Tversky. 1993. Cognitive Maps, Cognitive Collages and Spatial Mental Models. In Frank, A.U., Campari, I. (eds.) *Proceedings of the European Conference on Spatial Information Theory COSIT*, Elba, Italy. Springer-Verlag.

Webster. 1990. Rule-based Spatial Search. *International Journal Geographical Information Systems* 4 (3) : 241 - 260.

L. A. Zadeh. 1974. "Fuzzy Logic and Its Application to Approximate Reasoning". In *Information Processing*. : North-Holland Publishing Company.

E. C. Zeeman. 1962. "The Topology of the Brain and Visual Perception". In *The Topology of 3-Manifolds*. Edited by M. K. Fort. 240 -256. Englewood Cliffs, NJ: Prentice Hall.