Tolerance Geometry - Euclid's First Postulate for Points and Lines with Extension

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ABSTRACT

Object representation and reasoning in vector based geographic information systems (GIS) is based on Euclidean geometry. Euclidean geometry is built upon Euclid's first postulate, stating that two points uniquely determine a line. This postulate makes geometric constructions unambiguous and thereby provides the foundation for consistent geometric reasoning. It holds for exact coordinate points and lines, but is violated, if points and lines are allowed to have extension. As an example for a point that has extension consider a point feature that represents the city of Vienna in a small scale GIS map representation. Geometric constructions with such a point feature easily produce inconsistencies in the data. The present paper addresses the issue of consistency by formalizing Euclid's first postulate for geometric primitives that have extension.

We identify a list of six consequences from introducing extension: These are 'new qualities' that are not present in exact geometric reasoning, but must be taken into account when formalizing Euclid's first postulate for extended primitives. One important consequence is the positional tolerance of the incidence relation ("on"-relation). As another consequence, equality of geometric primitives becomes a matter of degree. To account for this fact, we propose a formalization of Euclid's first postulate in Łukasiewicz t-norm fuzzy logic. A model of the proposed formalization is given in the projective plane with elliptic metric. This is not a restriction, since the elliptic metric is locally Euclidean. We introduce graduated geometric reasoning with Rational Pavelka Logic as a means of approximating and propagating positional tolerance through the steps of a geometric construction process. Since the axioms (postulates) of geometry built upon one another, the proposed formalization of Euclid's first postulate provides one building block of a geometric calculus that accounts for positional tolerance in a consistent way.

The novel contribution of the paper is to define geometric reasoning with extended primitives as a calculus that propagates positional tolerance. Also new is the axiomatic approach to positional uncertainty and the associated consistency issue.

1. INTRODUCTION

In vector based geographic information systems (GIS) object representation is based on Euclidean geometry. Euclidean geometry relies on the idea that points are infinitely small. Points are the indivisible building blocks of geometric reasoning. In conflict with this idea is the fact that points in GIS map representations often represent geographic entities that in reality have extension. As an example, Figure 1 sketches two representations of the city of Vienna in different levels of detail (see Appendix): In Figure 1a, Vienna is represented by a single coordinate point, whereas in Figure 1b, its extended character is visible. A geometric construction that operates with the point representation of Vienna disregards its true extension, and consequently disregards the extension of the output. Existing heuristic solutions do not provide control over the behavior of a geometric construction w.r.t. extension. When plugged together, heuristics produce exceptions which must be treated separately. The present paper poses the question if it is possible to formalize geometric reasoning with points and lines that have extension *in a consistent way*.

We approach the issue of consistency by adopting an axiomatic standpoint of geometry. The paper provides a first step towards an answer of the above question by addressing the most fundamental axiom of all classical geometries, namely *Euclid's first postulate*. Euclid's first postulate states that the line determined by two points is unique. It makes geometric constructions unambiguous and thereby lays the foundation for consistent geometric reasoning.

It was shown in [23] positional tolerance plays a key role in geometric reasoning with extended primitives. An operator that connects two points with extension by a line with extension is either not practically useful in GIS, or introduces ambiguity into Euclid's first postulate: There is no law of nature telling the 'right' way of connecting Vienna with Munich by a linear entity. Despite this lack of principle, the generated ambiguities are not arbitrary: Based on some general assumptions on the nature of extended objects in a GIS context, we show that it is possible to derive a location constraint for the output object of a geometric construction. Since location constraints provide a certain amount of tolerance in positioning an object, we call the resulting ambiguity 'positional tolerance', and the respective formalism 'tolerance geometry'. The notion of 'tolerance geometry' was proposed by F.S. Roberts for any geometry whose primitives are obtained by "substituting closeness for identity" ([17], p.68).

As a consequence of introducing extension, six 'new qualities' emerge in connection with Euclid's first postulate: positional tolerance of incidence, graduation of equality of points and of lines, significance of size and distance, granularity, and weak transitivity of equality. We provide a formalization of Euclid's first postulate for extended primitives that accounts for these 'new qualities'. One way to define a graduated version of the equality relation is to define a reasonable fuzzy extension. The proposed formalization of Euclid's first postulate extends the Boolean formalization by translating it into fuzzy logic, and adapting it to fit the granularity component. Due to its metric properties, we choose Łukasiewicz t-norm fuzzy logic. We provide a model of the proposed fuzzy formalization in the projective plane with the elliptic metric. Since the elliptic plane is locally Euclidean, the proposed model is an approximation for Euclidean reasoning.

As a means of approximating and propagating positional tolerance, we introduce graduated geometric reasoning with Rational Pavelka Logic. Rational Pavelka Logic builds upon Łukasiewicz fuzzy logic, and allows for deriving a degree of equality of extended lines from Euclid's first postulate.

The remainder of the paper is structured as follows: Section 2 gives a review of related work. Section 3 discusses general assumptions on points and lines with extension, and provides a formal interpretation in the projective plane with elliptic metric. We derive a list of six 'new qualities' that result from introducing extended primitives into Euclid's first postulate. Section 4 develops formal interpretations of the incidence and equality relations for extended primitives that are based on the six new qualities. Section 5 gives a fuzzification of Euclid's first postulate in Łukasiewicz logic. Section 6 introduces approximate geometric reasoning with Rational Pavelka Logic, and exemplifies its application to the fuzzified Euclid's first postulate and tolerance propagation. Section 7 concludes with a summary of contributions and a discussion of further work.

2. RELATED WORK

Extended objects may be interpreted as location constraints. The issue of geometric reasoning with extended objects can be seen as a special case of geometric reasoning with positional tolerance. In the field of GIS, the concept of epsilon-tolerance has been used extensively. One of the oldest references addressing this topic dates back to 1966: J. Perkal [13] introduced the concept of epsilon-bands for empirical curves. Other references are, e.g., Pullar [16] and Christman et. al. [3], who address the problem of spurious objects and numerical inconsistencies caused by digital arithmetic. An example of a recent reference is E. Clementini [4], who gives a model for GIS line features that have extension. Clementini's model is built upon the model for regions with broad boundaries and allows for deriving topological relations between 'uncertain lines', but does not discuss geometrical operations. The different approaches improved on a multitude of practical problems like e.g. coincidence of points, line crossing, conflation, or data integration. Yet, a closed and consistent solution for geometric operations is still missing. The present work addresses the issue of consistency by adopting an axiomatic approach which extends the classical axioms of exact Euclidean geometry. This has three advantages: First, it is structurally close to exact geometry. Second, consistency of the calculus can be investigated with the tools of mathematical logic. Third, it allows for logical deduction and theorem proving.

In 1973, F.S. Roberts [17] developed an axiomatic 'tolerance geometry' to describe the indistinguishability of stimuli in visual perception. His work is based on Tarski's axiomatization of elementary geometry [21] for the simplest case of one-dimensional finite point sets. Positional tolerance is introduced by replacing the primitive relation of *betweenness* by the notion of ε*-betwenness*, which tolerates fixed but arbitrary positional errors smaller then ε. In 1980 M. Katz extended Robert's work to variable tolerance thresholds and infinite one-dimensional point sets [11]. Based on Goguen's *logic of inexactness* [8] Katz introduces *graduated equality* and *betweenness* relations. Instead of using Tarski's axiomatization of elementary geometry, the present approach is based on (a small part of) the axiomatization given by D. Hilbert [10]. While Tarski uses *points* as the only sort of objects, Hilbert follows the traditional approach of a two-sorted formulation based on *points* and *lines*. This is advantageous for GIS applications, where lines with positional tolerance are often initially given and not derived from points with positional tolerance.

In his 1989 doctoral thesis called 'fuzzy geometry' [15] T. Poston lays the foundations for a mathematics that is based on ε-points with crisp, i.e. non-graduated, thresholds. Several of the formal concepts used in the present paper are graduated versions of concepts introduced by Poston. Numerous axiomatic approaches exist, that aim at *restoring exact geometry* from primitives that have extension. The best known of these approaches is A. Tarski's Geometry of Solids [20]. A comprehensive overview of such 'point-free geometries' is given in [6]. In the GIS community B. Bennett's 'Region Based Geometry' [1] is of particular relevance. In contrast to these approaches, the present work aims at *approximating the behavior of exact geometry* with extended primitives.

In GIS, fuzzy set theory is used to *describe* geographic entities with unsharp boundaries (cf. e.g. [5]). As opposed to this, the present paper addresses crisp entities and uses mathematical t-norm fuzzy logic [9] to fuzzify geometric *reasoning*.

3. EXTENDED GEOMETRIC PRIMITIVES

Axiomatic geometry is an abstract logical *theory*, and as such is specified by a set of axioms. A *model* of a logical theory is an *interpretation* of its primitives in another domain, such that the axioms are satisfied. For example, the Cartesian model of Euclidean geometry provides an interpretation of the geometric primitives *point*, *line*, *equality*, *incidence*, *congruency*, etc. in the real plane $\mathbb{R}^2$, which is the domain of the interpretation. It can be shown that the Cartesian interpretation satisfies the Euclidean axioms. As a first step towards a geometric theory of extended points and lines, the present paper is concerned with only one axiom, namely Euclid's first postulate, which is a part of all classical geometries, in particular of Euclidean and projective geometry. It uses the primitive objects *point* and *line*, and the primitive relations *equality* and *incidence*.

In subsection 3.1 we propose an interpretation of the geometric primitives *point* and *line*, which captures an intuitive meaning of 'point with extension' and 'line with extension' from a GIS perspective. Subsections 3.2 - 3.7 investigate the primitive relations *equality* and *incidence* for 'points with extension' and 'lines with extension' w.r.t. Euclid's first postulate: Six 'new qualities' are identified that must be taken into account when defining an appropriate interpretation of *equality* and *incidence*. These new qualities are the following:

1) Incidence of 'points and lines with extension' has positional tolerance (subsection 3.2).

2) Equality of 'lines with extension' is graduated (subsection 3.3).
3) Equality of 'points with extension' is graduated (subsection 3.4).
4) Size and distance matter (subsection 3.5).
5) Extension introduces granularity (subsection 3.6).
6) Graduated equality is weakly transitive (subsection 3.7).

3.1 Points and Lines with Extension

In the present paper we are concerned with geographic entities that have extension in space, and whose GIS representations are intended to be used as primitive objects of a geometric construction. We refer to such entities as 'points with extension' and 'lines with extension'. This subsection gives a formal definition of these notions.

We confine our considerations to two dimensional object representations with sharp ('crisp') boundaries: For such objects, the indeterminacy of the object's boundary is negligible compared to its extension in space. Examples are parcels of land, buildings, countries, most lakes, roads, or streams. We do not consider point measurements like GPS coordinate points, which might have relevant measurement inaccuracy. We also do not consider objects with vague boundaries like mountains or pollution plumes.

We differentiate between an *actual* GIS object representation (e.g. in a user data set), and a *potential* representation as an extended object in an underlying metric space (e.g. in a reference data set). As shown in the introduction using the example of Vienna, the actual GIS object representation can be punctual, while potentially the objects' extension can be displayed, e.g. in a different level of detail. In other words, a metric space exists, where the object's extension can be represented.

For object representation in GIS it is common to use either Cartesian coordinates or homogeneous coordinates. Homogeneous coordinates are coordinates for the projective plane. The present paper uses the *spherical model of the projective plane* $\mathbb{P}^2$ as an interpretation domain for defining 'point with extension' and 'line with extension'. Before doing so, we give some preliminary definitions (cf. [2]):

Definition 1. The *spherical model of the projective plane* is defined as the three dimensional real unit sphere $\mathbb{S}^2 := \left\{ p \in \mathbb{R}^3 \mid \|p\| = 1 \right\}$ where antipodal points, $+p$ and $-p$ are identified, i.e. $\mathbb{P}^2 := \mathbb{S}^2 / \pm$. The elements of $\mathbb{P}^2$ are called *projective points*. *Projective lines* are the great circles of $\mathbb{S}^2$, where antipodal points are identified.

Definition 2. The *elliptic metric* on $\mathbb{P}^2$ is defined by

$$\varepsilon : \mathbb{P}^2 \to [0, \pi/2], \quad \varepsilon(p,q) := \operatorname{Arccos}|p \cdot q|, \tag{1}$$

where $p \cdot q = p_1 q_1 + p_2 q_2 + p_3 q_3$ denotes the standard inner product in $\mathbb{R}^3$.

The elliptic metric is a metric distance and the canonical metric in the projective plane. The elliptic distance between two projective points is the acute angle between corresponding $\mathbb{R}^3$-points on the unit sphere. Projective lines are isomorphic to a circle, and consequently are unbounded, but have finite length. Since antipodal points are identified, the elliptic length of an exact line is π, and the maximal elliptic distance of any two points is $\pi/2$. Locally, i.e. for small distances, the elliptic distance approximates the Euclidean distance. Every non-zero scaling of ε is also a metric distance on $\mathbb{P}^2$, and is also called an elliptic metric [2]. In the present paper, we use the elliptic metric $(2/\pi) \cdot \varepsilon$. Here, the maximum distance of any two points is 1. We call the projective plane, endowed with the elliptic metric $(2/\pi) \cdot \varepsilon$, the *elliptic plane*.

With the terms "point with extension" and "line with extension" we refer to an augmentation of the elliptic interpretation of projective geometry: A projective point in the elliptic plane can be viewed as a point on (one hemisphere of) the unit sphere. We can 'add extension' to the point by considering a topological neighborhood of the point. Here, the topology used is the topology induced by the elliptic metric. Using the duality of projective points and lines, we can 'add extension' to a projective line by considering a topological neighborhood of its dual projective point: In the spherical model of the projective plane, the dual projective point of a projective line is perpendicular to the great circle representing the projective line. It represents the line's parameters. In the homogeneous plane, these parameters can be interpreted as "parallel distance" and direction of the projective line. In other words, an extended elliptic line consists of projective lines whose parameters vary continuously.

Definition 3. An *extended elliptic point* is a non-empty bounded and regular closed subset of the elliptic plane. An *extended elliptic line* is a set of projective lines in the elliptic plane whose corresponding set of dual projective points is an extended elliptic point.

The reason for not using the Euclidean plane as an interpretation domain is that the Euclidean plane does not provide a canonical metric distance between Euclidean lines. Using parallel distance and direction as parameters is possible, but is not compatible with the Euclidean distance between Euclidean points.

In the remainder of the paper we use the term ′exact point (line)′ to denote a projective point (line), and the term ′extended point (line)′ to denote an extended elliptic point (line). With the term *point* (*line*), written in italic, we denote a variable of the logical theory, without assigning a specific interpretation.

3.2 Incidence of Extended Points and Lines has Positional Tolerance

In the last subsection, we gave an interpretation of the geometric primitives *point* and *line* that adds extension to the usual elliptic interpretation. The present subsection shows that an appropriate interpretation of the primitive relation of *incidence* should incorporate positional tolerance.

As an illustration, Figure 2 (see Appendix) sketches three typical geographic scenarios, where two extended objects can be classified as incident: Figure 2a shows a floating water mill that is installed in a stream. The polygon representing the mill can be interpreted as an extended point; the stream can be interpreted as (a segment of) an extended line. The polygon representing the mill overlaps with the polygon representing the stream. Figure 2b shows tracks of a city railway, and a station building, whose representation overlaps with the railway tracks. Figure 2c shows a marina on the banks of a stream. If the marina is aggregated to a single polygon, it overlaps the polygon representing the stream. These examples suggest that incidence of extended objects should be modeled by the *overlap* relation.

In exact geometry, incidence of a point and a line determines the position of the point up to one dimension along the line. The overlap relation does not determine the position of the extended point up to one dimension, but allows for some positional tolerance in both dimensions.

In the present paper we simplify the formalizing task by *limiting the scope of the incidence relation*: instead of considering an overlap relation, we model incidence by the *subset relation*. In other words, an extended line can not be incident with a point that is 'larger' than itself. In Figure 2, only example (a) of the floating water mill complies with this interpretation. In the sequel, we say "P is incident with Q", if the extended point P is a subset of the extended line Q.

3.3 Equality of Extended Lines is Graduated

In the exact model of geometry, Euclid's first postulate guarantees that two distinct exact points p and q uniquely determine an exact line. This is not the case for extended points and lines, where *incidence* is interpreted by the subset relation: Figure 3a (see Appendix) illustrates that if two distinct extended points P and Q are incident with two extended lines L_1 and L_2, then L_1 and L_2 are not necessarily equal in the sense of set equality. In other words, Euclid's first postulate does not apply if the primitive relation of *equality* is interpreted by set equality. To restore the validity of Euclid's first postulate, the interpretation of *equality* must be redefined.

P and Q do not uniquely determine an extended line. Yet, if two lines L_1 and L_2 are incident with P and Q, they are "closer together", i.e. "more equal" than arbitrary extended lines that are not constrained by incidence with P and Q. The further P and Q move apart from each other, the closer, i.e. "more equal", L_1 and L_2 become. One way to model this fact is to allow for *degrees of equality* of extended lines. Graduated equality should be an inverse distance measure.

3.4 Equality of Extended Points is Graduated

Graduated equality of extended lines compels graduated equality of extended points. As an example, Figure 3b sketches a situation where two extended lines L and L_1 intersect in an extended point P_1. Another extended line L_2 is close to L_1, i.e. L_1 and L_2 are "very equal". The extended point P_2 lies in the set-intersection of L_2 and L. Consequently, P_2 is closer to P_1 than most arbitrary extended points who are not constrained by L, L_1, and L_2. In other words, the "more equal" L_1 and L_2 are, the "more equal" are P_1 and P_2. We model this fact by allowing for degrees *of equality* of extended points. Again, graduated equality should be an inverse distance measure.

3.5 Size and Distance matter

In exact coordinate geometry, points and lines have discrete size and distance: A point or line has (infinitely small) size if it exists, and no size, if it does not exist. Two exact points or lines have distance, if they are distinct, and have no distance, if they are equal[1]. In case points and lines have extension, size and distance are measured continuously, and are real-valued. Their values influence the degree of equality of two extended points or lines.

As an example, Figure 4 (see Appendix) shows two extended points P and Q, and an extended line L that is incident with P and Q. The arrows indicate the two degrees of freedom that are induced by the incidence relation, namely direction (Figure 4a) and "parallel distance" (Figure 4b). Using the example of the directional parameter, Figure 4c illustrates that the positional tolerance of L increases with the size of L. Consequently, the (minimal possible) degree of equality of two incident extended lines decreases with their sizes. Similarly it can be seen that it increases with increasing sizes of P and Q. Figure 4d shows that the (minimal possible) degree of equality of two incident extended lines decreases with the distance of P and Q.

3.6 Extension Introduces Granularity

In exact coordinate geometry, distance is measured discretely. As a consequence, two distinct coordinate points p and q determine a coordinate line uniquely, even if p and q are arbitrarily close to one another. This is not necessarily the case for distinct extended points P and Q. To see this, imagine that the extended points P and Q in Figure 3a move closer together: If P and Q are "very close" to one another and the extended line L is "too broad", then it may happen that P and Q behave like one single point with respect to L.

Figure 5a (see Appendix) illustrates this case: Despite the fact that P and Q are distinct extended points that are both incident with L, they do not specify any *directional constraint* for L. Consequently, the difference between the directional parameters of two incident lines may assume its maximum (90°), like it is the case for L and L' in Figure 5a. Since we measure graduated equality of extended lines inverse to distance, and since the distance between extended lines depends on the two parameters parallel distance and direction, L and L' in Figure 5a have *maximum* distance. In other words, their degree of equality is zero, even though they are distinct and incident with P and Q.

The above observation can be interpreted as *granularity*. Note that in this context grain size is not constant, but depends on the sizes of the involved extended objects.

3.7 Graduated Equality is Weakly Transitive

As stated in subsections 3.3 and 3.4, it is reasonable to assume that graduated equality is inverse to distance. As a consequence, graduated equality is not transitive in the classical sense: Figure 5b sketches three equidistant extended lines L_1, L_2 and L_3. Since the pairs L_1, L_2 and L_2, L_3 are equidistant, they should be

[1] This is the *discrete metric* Δ, defined by $\Delta(p,q)=1$, if $p \neq q$, and $\Delta(p,q)=0$, if $p=q$.

assigned the same degree of equality λ. Yet, we must not conclude that L_1 and L_3 are equal to degree λ, since they have larger distance.

Yet, we can not afford to drop transitivity completely (cf. [25]): In the setting of Euclid's first postulate shown in Figure 6a (see Appendix), the only relation between the extended lines L_1 and L_2 is established by the fact that both are incident with *P* and *Q*. *This can be interpreted as a weak form of transitivity.* To see this, define

$$\overline{PQ} := \{l_{(p,q)} \mid p \in P, q \in Q, p \neq q\}, \tag{2}$$

where $l_{(p,q)}$ is the unique exact line connecting the exact points *p* and *q* (Figure 6b). Assume that L_1 is equal to $\overline{PQ}$ with degree λ_1 (Figure 6c), that $\overline{PQ}$ is equal to L_2 with degree λ_2 (Figure 6d), and that L_1 is equal to L_2 with degree μ. The equality degree μ is in general larger than the equality degree of two arbitrary extended lines, since arbitrary extended lines can have maximum distance (zero equality). In contrast to this, μ depends on *P* and *Q*, and consequently on λ_1 and λ_2. We will show in subsection 4.2.4 how this relationship can be formalized.

The phenomenon is a graduated version of the *Poincaré paradox* [7], which is named after the famous French mathematician and theoretical physicist Henri Poincaré. Poincaré repeatedly pointed out that equality of sensations and measurements are in many cases intransitive (cf. e.g. [14]).

4. EQUALITY AND INCIDENCE OF EXTENDED POINTS AND LINES

Subsection 3.1 gave an interpretation of the primitive objects *point* and *line* that adds extension to the usual elliptic interpretation. In the present section, we develop interpretations of the primitive relations *equality* and *incidence* for extended points and lines based on the six 'new qualities' identified in subsections 3.2 - 3.7.

4.1 Incidence of Extended Points and Lines

As stated in subsection 3.2, it is reasonable to interpret incidence of extended points and lines by the subset relation.

Definition 4: The *incidence* relation between an extended point *P* and an extended line *L* is defined by

$$on(P,L) := (P \subseteq L) \quad \in \{0,1\}, \tag{3}$$

where the subset relation $\subseteq$ refers to *P* and *L* as subsets of the elliptic plane.

The incidence relation (3) is a Boolean relation, assuming either the truth value 1 (*true*) or the truth value 0 (*false*). Since a Boolean relation is a special case of a graduated relation, i.e. since $\{0,1\} \subset [0,1]$, we will be able to use relation (3) as part of a fuzzy theory later on.

4.2 Equality of Extended Points and Lines

As stated in subsections 3.3 and 3.4, it is reasonable to interpret equality of extended points (lines) by a graduated relation that is inverse proportional to their distance. We represent graduated equality as a logical predicate in Łukasiewicz fuzzy logic, since Łukasiewicz fuzzy logic bears a strong connection to metric distance. Subsection 4.2.1 briefly introduces the operators and some properties of Łukasiewicz logic, which we will use in the present section and in sections 5 and 6. Subsections 4.2.2 - 4.2.4 establish the connection with metric distance and show how to account for weak transitivity introduced in subsection 3.7. Subsection 4.2.5 derives a set of boundary conditions that are necessary to incorporate granularity (cf. section 3.6). Finally, subsection 4.2.6. defines an interpretation of equality of extended points that complies with these granularity conditions.

4.2.1 Łukasiewicz Logic

Łukasiewicz logic is one of the three fundamental t-norm fuzzy logics. In t-norm fuzzy logics, a *triangular norm (t-norm)* plays the role of a graduated conjunction operator. A t-norm is a binary operation $*:[0,1]^2 \to [0,1]$ that is commutative, associative, non-decreasing, and has 1 as its unit element [9]. The *Łukasiewicz t-norm* $\otimes$, its residuated implication $\to$, and the corresponding negation $\neg$ are given by

$$x \otimes y = \max\{x + y - 1, 0\}, \tag{4}$$

$$x \to y = \begin{cases} 1 & \text{for } x \leq y \\ 1 - x + y & \text{for } x > y \end{cases}, \tag{5}$$

$$(\neg x) = (1 - x), \tag{6}$$

respectively. In section 6.2, we will need the following equivalences:

$$[x \to y] = 1 \iff x \leq y, \tag{7}$$

$$[(x \otimes z) \to y] = 1 \iff [z \to (x \to y)] = 1. \tag{8}$$

4.2.2 Pseudometric Distance

A pseudometric distance, or *pseudometric*, is a map $d: M^2 \to \mathbb{R}^+$ from a domain M into the positive real numbers (including zero), which is minimal, symmetric, and satisfies the triangle inequality:

$$d(x,x) = 0 \tag{9}$$

$$d(x,y) = d(y,x) \tag{10}$$

$$d(x,y) + d(y,z) \geq d(x,z). \tag{11}$$

d is called a *metric*, if additionally separability holds:

$$d(x,y) = 0 \iff x = y. \tag{12}$$

Well known examples of metric distances are the Euclidean distance, or the Manhattan distance. Another example is the elliptic metric for the projective plane defined in (1).

4.2.3 Fuzzy Equivalence Relations

The concept of a pseudometric distance is dual to the concept of a *Łukasiewicz fuzzy equivalence relation*: A fuzzy equivalence relation w.r.t. the Łukasiewicz t-norm $\otimes$ is a fuzzy relation $e: M^2 \to [0,1]$ on a domain M, which is reflexive, symmetric and $\otimes$-transitive:

$$e(x,x) = 1 \tag{13}$$

$$e(x,y) = e(y,x) \tag{14}$$

$$e(x,y) \otimes e(y,z) \le e(x,z). \tag{15}$$

e is called a *fuzzy equality relation*, if additionally separability holds:

$$e(x,y) = 1 \iff x = y. \tag{16}$$

If d is a pseudometric distance, then

$$e(x,y) := \max\{1 - d(x,y), 0\} \tag{17}$$

is a fuzzy equivalence relation w.r.t. $\otimes$ [22]. In case the size of the domain M is normalized to 1, equation (17) simplifies to

$$e(x,y) := 1 - d(x,y). \tag{18}$$

In other words, given a metric distance d on a normalized domain, equation (18) defines a graduated equality relation e by simple Łukasiewicz negation.

4.2.4 Approximate Fuzzy Equivalence Relations

As stated in subsection 3.7, a graduated equality relation between extended points or lines should be modeled by a weak form of transitivity. G. Gerla [7] shows that weak transitivity can be formalized by adding to the classical transitivity axiom (14) a graduated transitivity measure $trans: M \to [0,1]$:

$$e(x,y) \otimes e(y,z) \otimes trans(y) \le e(x,z)\ . \tag{19}$$

Here, *trans*(y) is a lower bound to the degree of transitivity that is permitted by y, i.e.

$$trans(y) \le \inf\{e(x,y) \otimes e(y,z) \to e(x,z) \mid x,y \in M\}\ . \tag{20}$$

A pair (*e,trans*) that is reflexive (13), symmetric (14), and weakly transitive (19) is called an *approximate fuzzy* $\otimes$*-equivalence relation*[2].

The concept of an approximate fuzzy $\otimes$-equivalence relation is dual to the concept of a *pointless pseudometric space* (δ, s) [7]:

$$\delta(x,x) = 0, \tag{21}$$

$$\delta(x,y) = \delta(y,x), \tag{22}$$

$$\delta(x,y) + \delta(y,z) + s(y) \ge \delta(x,z). \tag{23}$$

[2] Gerla uses the name *approximate similarity relation*. In the present abstract we use the name *approximate fuzzy equivalence relation* to stress the connection with the Boolean equality relation used in Euclid's first postulate.

Here, $\delta: M \to \mathbb{R}^+$ is a (not necessarily metric) distance between sets, and $s: M \to \mathbb{R}^+$ is a size measure. More specifically, $s(y)$ is an upper bound to the size of y. Inequality (23) is a weak form of the triangle inequality (5). It corresponds to the weak transitivity (19) of the approximate fuzzy $\otimes$-equivalence relation e. In case the size of the domain M is normalized to 1, e and *trans* can be represented by

$$e(x,y) := 1 - \delta(x,y), \quad trans(y) := 1 - s(y) \tag{24}$$

[7]. In other words, given a pointless pseudometric (δ, s) for extended regions on a normalized domain, equations (24) define an approximate fuzzy $\otimes$-equivalence relation (*e,trans*) by simple Łukasiewicz negation.

The extended points in the elliptic plane define a pointless pseudometric space, if we define (δ, s) as follows:

$$\delta(P,Q) := \inf\{d(p,q) \mid p \in P, q \in Q\}, \tag{25}$$

$$s(P) := \sup\{d(p,q) \mid p,q \in P\}, \tag{26}$$

where $d := [(2/\pi) \cdot \varepsilon] : \mathbb{P}^2 \to [0,1]$ is the scaled elliptic metric defined in section 3.1. The extended lines in the elliptic plane define a pointless pseudometric space, if we define (δ, s) by

$$\delta(L_1, L_2) := \inf\{d(l_1', l_2') \mid l_1 \in L_1, l_2 \in L_2\}, \tag{27}$$

$$s(L) := \sup\{d(l_1', l_2') \mid l_1, l_2 \in L\}, \tag{28}$$

where l', m' denote the dual points of the projective lines l, m.

4.2.5 Boundary Conditions for Granularity

As discussed in section 3.6 granularity enters Euclid's first postulate, if points and lines have extension: If two extended points P and Q are "too close" and the extended line L is "too broad", then P and Q behave like one single point w.r.t. the positional tolerance of L induced by the incidence relation. In other words, the degree of equality of P and Q depends not only on their distance - which we defined in (25) -, but also on their sizes (26) and on the size of L (28). For this reason we denote the graduated equality relation of P and Q w.r.t. L by $e(P,Q)[L]$, where L is included as an additional parameter.

Any interpretation of the graduated equality predicate $e(P,Q)[L]$ should satisfy the following three boundary conditions:

1. If $s(L) \ge \delta(P,Q) + s(P) + s(Q)$, then the positional tolerance of L is not constrained by the incidence relation (cf. Figure 5a), i.e. P and Q are fully equal w.r.t. L: $e(P,Q)[L] = 1$.
2. If $s(L) \le \delta(P,Q) + s(P) + s(Q)$, then the incidence relation imposes *some* constraint on the positional tolerance of L, but in general does not fix the location of L uniquely. The degree of equality of P and Q w.r.t. L should lie between zero and one: $0 < e(P,Q)[L] < 1$.
3. If $s(L) \le \delta(P,Q) + s(P) + s(Q)$ and $P = p$, $Q = q$ and $L = l$ are exact, then $s(L) = s(P) = s(Q) = 0$. Consequently, P and

Q determine the location of L uniquely, and we measure the equality of P and Q by the discrete metric: $e(P,Q)[L]=0$.

4.2.6 An Elliptic Model of Granulated Equality

From the boundary conditions on granularity proposed in the forgoing subsection, we can derive an interpretation of granulated graduated equality in the elliptic plane. Unfortunately the derivation is too long for presentation in the present paper. Instead, it is available on the authors' website [24].

Definition 5. The *equality* relation for extended points P, Q w.r.t. an extended line L, is defined by

$$e(P,Q)[L] := (2/\pi)\, \min\left\{ \mathrm{Arcsin}\left(H(P,Q)[L]\right),\ 1 \right\} \tag{29}$$

where

$$H(P,Q)[L] := \begin{cases} h(P,Q)[L], & \text{if } \ \delta(P,Q)+s(P)+s(Q) \neq 0, \\ \infty \quad , & \text{if } \ \delta(P,Q)+s(P)+s(Q) = 0, \end{cases} \tag{30}$$

and

$$h(P,Q)[L] := \frac{\tan\left[(\pi/4)\cdot s(L)\right]}{\tan\left[(\pi/4)\cdot\left(\delta(P,Q)+s(P)+s(Q)\right)\right]} . \tag{31}$$

Here, $\delta(P,Q)$, $s(P)$, $s(Q)$ and $s(L)$ are defined in (25), (26) and (28).

5. A FUZZIFICATION OF EUCLID'S FIRST POSTULATE

In sections 3 and 4 we gave interpretations of the geometric primitives *point*, *line*, *incidence*, and *equality*, which add extension to the usual elliptic interpretation. The present section shows that the proposed interpretation satisfies an adapted version of Euclid's first postulate. An adaptation is necessary, since

1. the classical (exact) version of Euclid's first postulate is formulated in Boolean logic, whereas the proposed interpretation uses fuzzy, i.e. multi-valued, relations.
2. Adding extension to exact points and lines introduces *size* as an additional parameter, which enters Euclid's first postulate in the form of the transitivity degree *trans*.

To formulate a fuzzified version of Euclid's first postulate, we first split the postulate

"Two distinct points determine a line uniquely." (32)

in two subsentences:

"Given two distinct points, at least one line exists that passes through them. " (33)

"If more than one line passes through two distinct points, then these lines are equal." (34)

A formalization in Boolean predicate logic reads as follows:

$$\neg\left[p=q\right] \ \rightarrow\ \exists l\ \left[on(p,l)\wedge on(q,l)\right], \tag{35}$$

$$\begin{aligned} \neg\left[p=q\right] \wedge \left[on(p,l_1)\ \wedge\ on(q,l_1)\right] & \\ \wedge\left[on(p,l_2)\ \wedge\ on(q,l_2)\right] &\ \rightarrow\ (l_1=l_2) \end{aligned} \tag{36}$$

Here, p,q stand for *points*, and l_1,l_2 stand for *lines*. The Boolean predicates = and *on* refer to *equality* and *incidence*. $\exists$ denotes the existential quantifier, $\neg,\wedge,\rightarrow$ stand for Boolean negation, conjunction, and implication, respectively.

A verbatim translation of (35) and (36) in the language of Łukasiewicz fuzzy logic yields

$$\neg e(P,Q) \rightarrow \sup_L \left[on(P,L)\otimes on(Q,L)\right], \tag{37}$$

$$\begin{aligned} \neg e(P,Q) \otimes\ & \left[on(P,L_1)\otimes on(Q,L_1)\right] \\ & \otimes\ \left[on(P,L_2)\otimes on(Q,L_2)\right]\ \rightarrow\ e(L_1,L_2), \end{aligned} \tag{38}$$

where P, Q denote *points*, L_1,L_2 denote *lines*. The symbol e replaces the symbol = for *equality*, since = is usually reserved to denote Boolean equality, *on* denotes *incidence*. The Łukasiewicz conjunction $\otimes$, negation $\neg$, and implication $\rightarrow$ have been introduced in section 4.2.1. The existential quantifier is replaced by the supremum operator *sup* [9].

The translated existence property (37) can be adopted as it is, since we define *on* as a Boolean relation (cf. Definition 4). The translated uniqueness property (38) must be adapted to the graduated and granulated version of equality given in Definition 5: We replace the term $\neg e(P,Q)$ on the left hand side of (38) by two terms, $\neg e(P,Q)[L_1]$ and $\neg e(P,Q)[L_2]$, one for each *line*, L_1 and L_2 , respectively. Since weak transitivity is used to relate the equality degrees of L_1 and L_2 via P and Q, the transitivity measure $trans(\overline{PQ})$ must be added:

$$\begin{aligned} & \left[\neg e(P,Q)[L_1]\ \otimes\ \neg e(P,Q)[L_2]\ \otimes\ trans(\overline{PQ})\right] \\ & \qquad \otimes\ \left[on(P,L_1)\ \otimes\ on(Q,L_1)\ \right] \\ & \qquad \otimes\ \left[on(P,L_2)\ \otimes\ on(Q,L_2)\right]\ \rightarrow\ e(L_1,L_2). \end{aligned} \tag{39}$$

It is possible to prove that the elliptic interpretations of incidence given by (3), graduated equality of extended lines given by (24) and (27), and granulated graduated equality of extended points given by (29) define a model of the proposed axioms (37) and (39). In other words, axioms (37) and (39) are satisfied with truth degree 1. Unfortunately, the length restriction of the present paper format does not allow for presenting it here. Instead, we provide the proof on the author's website [24].

6. GRADUATED GEOMETRIC REASONING

In this last section we address the question of how the adapted version of Euclid's first postulate given in (37) and (39) can be used in practical applications. Section 6.1 introduces *Rational Pavelka Logic* (RPL) as a tool for approximate reasoning. Section 6.2 shows how RPL can be used to propagate positional tolerance through the steps of a geometric construction.

6.1 Rational Pavelka Logic

As stated in section 4.2, one reason for choosing Łukasiewicz logic is its strong connection to metric spaces. Another reason is that Łukasiewicz logic can be extended to allow for graduated deduction rules in the style of J. Pavelka ([9], [12]). The according logic is called *Rational Pavelka Logic* (RPL). In RPL, *graduated deduction rules* allow for inferring partially true conclusions from partially true assumptions. This is useful in our setting: We can interpret an equality of degree $\lambda \in [0,1]$ of two extended lines L_1 and L_2 as the truth degree of the proposition "L_1 and L_2 are equal".

As the most important deduction rule, we address *Modus Ponens* (MP), which, in the classical case, is given by

$$MP: \frac{A, A \to B}{B}. \tag{40}$$

MP says that if the formulas A and $A \to B$ have truth degree 1, then formula B has truth degree 1. The *Graduated Modus Ponens* rule (GMP) of RPL is given by

$$GMP: \frac{(A, \lambda_A), (A \to B, \lambda_{A\to B})}{(B, [\lambda_A \otimes \lambda_{A\to B}])}. \tag{41}$$

Here, a pair (X, λ_X) is called *graduated formula*: X is a *syntactic* formula, and $\lambda_X \in [0,1]$ is a truth degree. GMP says that if the formula A has truth degree λ_A, and if formula $A \to B$ has truth degree $\lambda_{A\to B}$, then formula B has *at least* truth degree $\lambda_A \otimes \lambda_{A\to B}$. In other words, GMP derives a *lower bound approximation* of the real truth degree of B. $\lambda_A \otimes \lambda_{A\to B}$ is called a *deduced truth degree*.

In order to distinguish real and deduced truth degree, we denote the real truth degree of a syntactic formula B by $B = \lambda_B$, and a deduced truth degree for B by $\underline{\lambda}_B$. If a formula X is the result of repeated application of GMP, then its deduced truth degree $\underline{\lambda}_X$ can be interpreted as a *proof* of the fact that $X = \lambda_X \geq \underline{\lambda}_X$. In RPL, different proofs can yield different deduced truth degrees for the same formula X. Consequently, $\underline{\lambda}_X = 0$ only means that *no information* on the truth degree of X *can be deduced from the given facts*.

6.2 Euclid's First Postulate and Graduated Modus Ponens

If two extended points P and Q are incident with the extended lines L_1 and L_2, we can use GMP to derive a lower bound approximation for the equality of L_1 and L_2 from axiom (39).

Before doing the deduction, we simplify axiom (39): If P and Q are incident with L_1 and L_2, the Boolean incidence predicate defined in (3) has truth value 1 for all instances:

$$on(P, L_1) = on(Q, L_1) = on(P, L_2) = on(Q, L_2) = 1. \tag{42}$$

Since 1 is the unit of the t-norm, $x \otimes 1 = 1$ holds for all $x \in [0,1]$, and inserting (42) into (39) yields

$$\left[\neg e(P,Q)[L_1] \otimes \neg e(P,Q)[L_2] \otimes trans(\overline{PQ})\right] \to e(L_1, L_2). \tag{43}$$

Using the equivalence (8) introduced in section 4.2.1, we can rewrite (43) as series of implications:

$$trans(\overline{PQ}) \to \left[\neg e(P,Q)[L_2] \to \left[\neg e(P,Q)[L_1] \to e(L_1, L_2)\right]\right]. \tag{44}$$

If the sizes $s(P)$, $s(Q)$, and the set distance $\delta(P,Q)$ of P and Q are known, we can apply GMP successively to the graduated formulas $\left(trans(\overline{PQ}), \lambda_{\mathrm{trans}(\overline{PQ})}\right)$, $\left(e(P,Q)[L_2], \lambda_{\mathrm{e(P,Q)[L_2]}}\right)$, $\left(e(P,Q)[L_1], \lambda_{\mathrm{e(P,Q)[L_1]}}\right)$, and (44). Here, the truth degree $\lambda_{\mathrm{trans}(\overline{PQ})}$ must be estimated from $s(P)$ and $s(Q)$. The truth degrees $\lambda_{\mathrm{e(P,Q)[L_2]}}$ and $\lambda_{\mathrm{e(P,Q)[L_1]}}$ can be calculated from formula (29). The truth degree of (44) equals 1: Since (44) is a simplified version of axiom (39), it is satisfied by the interpretations of *point*, *line*, *incidence* and *equality* introduced in sections 3.1, 4.1, and 4.2.6 (cf. section 5).

The result of the deduction is the deduced truth degree

$$\underline{\lambda}_{\mathrm{e(L_1,L_2)}} = \lambda_{\mathrm{e(P,Q)[L_1]}} \otimes \lambda_{\mathrm{e(P,Q)[L_2]}} \otimes \lambda_{\mathrm{trans}(\overline{PQ})}, \tag{45}$$

which is a lower bound approximation of the real truth degree $e(L_1, L_2) = \lambda_{\mathrm{e(L_1,L_2)}}$, i.e. for the *degree of equality* of L_1 and L_2.

The approximation $\underline{\lambda}_{\mathrm{e(L_1,L_2)}}$ is an inverse measure of the *positional tolerance* of a "connection" of P and Q. This can be seen as follows: If L_1 and L_2 have the same size s_0, they can be interpreted as different positions (locations) of the same extended line. Since the real equality degree $e(L_1, L_2) = 1 - \delta(L_1, L_2)$ is inverse to the distance of L_1, L_2, cf. (24),

$$\delta(L_1, L_2) = 1 - e(L_1, L_2) = 1 - \lambda_{\mathrm{e(L_1,L_2)}} \leq 1 - \underline{\lambda}_{\mathrm{e(L_1,L_2)}} \tag{46}$$

holds for all extended lines L_1, L_2 of size s_0 that are incident with P and Q. In other words, $\underline{\lambda}_{\delta(\mathrm{L_1,L_2})} := 1 - \underline{\lambda}_{\mathrm{e(L_1,L_2)}}$ is an upper bound approximation for the *positional tolerance* of an s_0-sized "connection" of P and Q.

Axioms (37) and (39) for the adapted version of Euclid's first postulate can be summarized as follows:

$$\text{"Two distinct extended points determine an extended line up to positional tolerance."} \tag{47}$$

In practical application, a warning can be given, if the approximated positional tolerance of a construction exceeds a predefined threshold. In such a case, the real positional tolerance value does not necessarily exceed the threshold. In case exact knowledge about the involved objects is available, the exact positional tolerance can be calculated from exact shapes and locations. This can be done automatically, without user involvement. Only if the real positional tolerance value exceeds the threshold must the user be informed that the construction is ill-defined.

If no exact knowledge about shape and location is available, the procedure can be used to estimate whether a geometric construction is (more or less) well defined. Such a situation may arise in the context of ubiquitous computing, where the ability to represent and query textual descriptions of spatial configurations becomes increasingly important. The lack of detailed information on geographical entities may be caused by a limited bandwidth of hand held devices or by incomplete information from a participatory database.

7. CONCLUSIONS AND FUTURE WORK

The paper discusses a special form of positional uncertainty in vector based GIS, namely positional tolerance that arises from geometric constructions with extended primitives. We propose a framework for approximating and propagating positional tolerance through the steps of a geometric construction. As a first step towards this goal, we address Euclid's first postulate, which lays the foundation for consistent geometric reasoning in all classical geometries. We identified six qualities that are not present in the Boolean version of Euclid's first postulate, but must be taken into account when introducing extension to the primitives. We proposed a formal interpretation of the geometric primitives *points*, *line*, *incidence* and *equality*, which incorporates an intuitive meaning of two dimensional extension. We proposed a fuzzification of the axioms of Euclid's first postulate in Łukasiewicz logic, together with a proof on the author's website, which shows that the proposed interpretations constitute a valid model of the fuzzified axioms. We introduced approximate geometric reasoning with Rational Pavelka logic as a means of propagating positional tolerance through the steps of a geometric construction.

We currently implement the proposed elliptic model in the functional programming language HASKELL. Euclidean data is transformed into homogeneous coordinates. After calculating the equality values in the elliptic model, a local Euclidean approximation is visualized in the $z=1$ plane. After an initial testing phase with artificial data, real GIS data will be used to test for the practical applicability of the model.

The current approach interprets *incidence* by the Boolean *subset* relation. In order to provide a more realistic interpretation, two extensions are necessary: In a first step, the Boolean *subset* relation should be replaced by the Boolean *overlap* relation. In a second step, a graduated incidence relation could be implemented that is inverse to the orthogonal distance of an extended point to an extended line.

It is our objective to extend the proposed axiomatic calculus by further geometric primitives like, e.g., the *betweenness* relation, and the respective axioms.

8. ACKNOWLEDGEMENTS

The idea for the present work emerged from a keynote speech given by Lotfi A. Zadeh about granular computing [26]. In constrast to the present approach, Zadeh suggested an unprecisiated geometric calculus, which allows for multiple modalities of uncertainty in position.

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10. APPENDIX

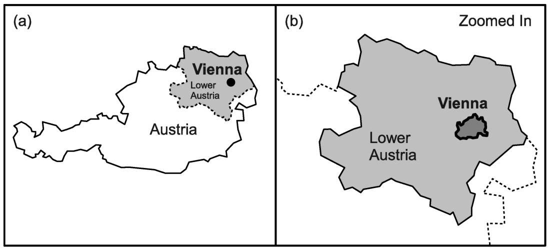

Figure 1. Vienna, (a) represented by a coordinate point, and (b) represented as an extended geographic entity.

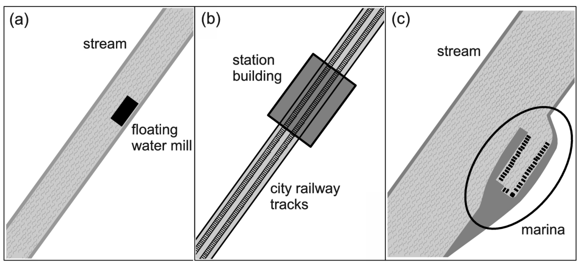

Figure 2. Three examples of extended objects that are incident with an extended line.

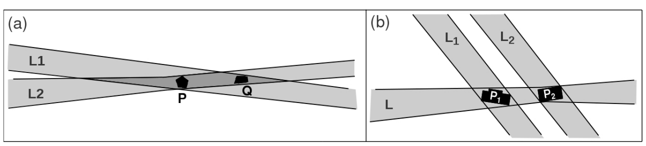

Figure 3. (a) Two extended points do not uniquely determine the location of an incident extended line. (b) Graduated equality of extended lines compels graduated equality of extended points.

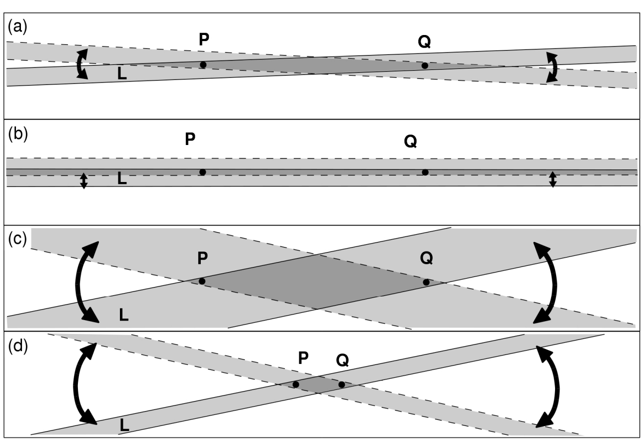

Figure 4. Size and distance matter.

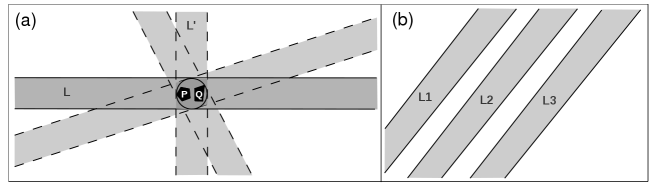

Figure 5. (a) *P* and *Q* are indiscernible for *L*. (b) Graduated equality is not transitive in the classical sense.

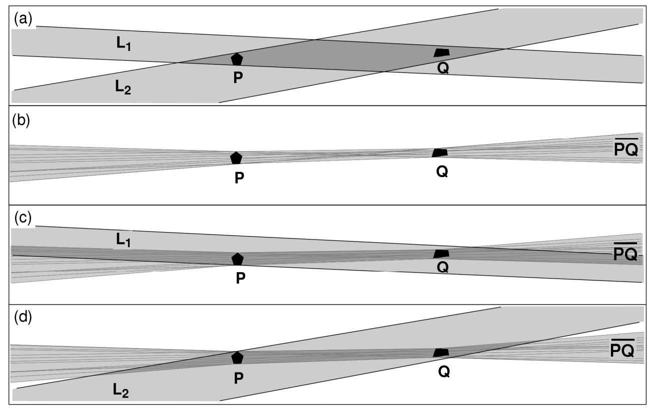

Figure 6. (a) L_1 and L_2 are incident with *P* and *Q*. (b)-(d) $\overline{PQ}$ establishes a spatial relationship between L_1 and L_2.